PREFERENCE UNDER AMBIGUITY: TESTING AND IDENTIFICATION

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Contents

List of Tables

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Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself, except Chapter 2 and Chapter 3, each of which has derived from collaborative work with both Herakles Polemarchakis and Larry Selden. It has not been submitted in any previous application for any degree.

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Abstract

Preference under Ambiguity: Testing and Identification

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This dissertation focuses on testing and identifying individual ambiguity preference under the framework of "smooth ambiguity preference" developed by Klibanoff, Marinacci, and Mukerji (2005). Following the seminal contributions of Allais (1953) and Ellsberg (1961), experimental data have consistently demonstrated that individuals do not behave in accordance with predictions of the expected utility model when they face uncertainty. As one important class of ambiguity utility, the smooth ambiguity model distinguishes ambiguity aversion from risk aversion, which makes the comparative statics possible. However, currently there is little work on testing and recovering such preferences based on observable choices.

The dissertation contains four parts. Chapter 2 uses two approaches to derive the necessary and sufficient conditions for observed individual portfolio choice to be compatible with the smooth ambiguity preference. The first approach is the revealed preference method, and is based on finite observations. The second approach is demand function testing, and is based on infinite observations. Chapter 3 establishes the conditions under which the smooth ambiguity preference can be uniquely identified from individual demand functions. In Chapter 4, I extend the argument of Varian (1988) to multiple observations and incomplete market case to non-parametrically test different shapes of risk aversion, and then to test hypotheses on shapes of ambiguity aversion. In Chapter 5, to use household survey data to identify household risk and ambiguity aversion, I build a simple parametric model to identify household risk and ambiguity aversion from their saving and portfolio choice. The data from the Bank of Italy Survey on Household Income and Wealth 2008 and 2010 support the constant relative risk aversion and constant relative ambiguity aversion hypothesis, and give evidence of the magnitude of household risk and ambiguity aversion.

Chapter 1

Preference under ambiguity

1.1 Testing and identification

Over the past 50 years following the seminal contributions of Allais (1953) and Ellsberg (1961), laboratory data predominantly based on observed choices over lotteries, have consistently demonstrated that individuals do not behave in accordance with predictions of the expected utility model. In recent years, extensive effort has concentrated on developing models that accommodate the fact that in many real world problems individuals face choices characterized by uncertainty, and not just risk. This is not handled well by expected utility maximizers who exhibit behaviour reflecting the Ellsberg paradox. One class of models that focus on incorporating uncertainty and overcoming the problems of the expected utility theory is referred to as ambiguity preferences.

A number of alternative models of ambiguity preferences have been developed. These include multiple-priors or maxmin preference, Gilboa and Schmeidler (1989), smooth ambiguity preference, Klibanoff, Marinacci, and Mukerji (2005), multiplier preference, Anderson, Hansen, and Sargent (2003), variational preference, Maccheroni, Marinacci, and Rustichini (2006), vector expected utility, Siniscalchi (2009). For some relevant literature on decision theory, see Etner, Jeleva, and Tallon (2012), Gilboa (2009), and Gilboa and Marinacci (2011). The applications of ambiguity preference in economic and finance modelling include portfolio choice and asset pricing Collard, Mukerji, Sheppard, and Tallon (2011), Epstein and Schneider (2010), Gollier (2011), Guidolin and Rinaldi (2013), Ju and

Miao (2012), and macroeconomics Hansen and Sargent (2001), Hansen and Sargent (2007), Hansen, Sargent, Turmuhambetova, and Williams (2006).

This dissertation focuses on the "smooth ambiguity preference" model due to Klibanoff, Marinacci, and Mukerji (2005). There are two reasons: first of all, this model explicitly separates ambiguity aversion from risk aversion, so it is meaningful to test the restrictions of ambiguity aversion, and to identify the ambiguity aversion index; secondly, the utility function is well-behaved, i.e. it is differentiable and concave, so the analytic machinery developed in utility and expected utility theory can be generalized to deal with ambiguity utility.

This dissertation will address two questions: what are the testable restrictions of the smooth ambiguity model on observable individual portfolio choice? if an individual's portfolio choice is compatible with the smooth ambiguity model, could his ambiguity preference be uniquely identified? The first question deals with existence problem, and the second deals with uniqueness problem.

In Chapter 2, we will derive the necessary and sufficient conditions for observed portfolio choice to be consistent with the strictly increasing, and strictly concave smooth ambiguity preference, using two approaches: revealed preference approach and demand function approach. The revealed preference approach developed by Afriat (1967) and Varian (1982) requires only finite observations of individual portfolio choice, and does not specify any parametric utility functional form, thus is fully nonparametric. The conditions developed here consist of nonlinear inequalities, and can be used to test portfolio choice from incomplete markets. The demand function approach assumes individual asset demand functions are given. We develop two tests based on demand functions: a functional form test and a demand derivative test. The demand functional form test requires the observed asset demands have particular functional form restrictions, while the demand derivative test gives Slutsky−like restrictions. Unlike in the revealed preference test, we assume there are complete state consumption claims contingent on both ambiguity states and risk states. Such an assumption is extreme, and such data can be obtained only under well controlled laboratory experiments. However, we think to derive theses restrictions in ideal data case is an important first step to understanding the implications of smooth ambiguity model.

In Chapter 3, we assume observed individual asset demands pass the tests in Chapter 2, and develop conditions under which his smooth ambiguity utility function can be uniquely identified. In Chapter 2, the smooth ambiguity preference can be explicitly constructed if an individual's asset demands satisfy the revealed preference conditions; however, such construction is not unique, since finite observations cannot pin down an individual's indifference curves. We tackle uniqueness by assuming individual asset demand functions given. This work builds on the recoverability literature for expected utility preferences in Green, Lau, and Polemarchakis (1979) and Dybvig and Polemarchakis (1981). For us to address the recoverability of ambiguity preferences, several important new technical results are required. We assume there is one riskless asset and one ambiguity-free asset (i.e. the payoff distribution is invariant across ambiguity states). We show that under these assumptions and one technical condition (full rank condition), the individual's risk aversion index and ambiguity aversion index can be uniquely identified from his asset demand functions. The technical condition basically means individual has ambiguity on the mean return of risky assets. The existence of riskless and ambiguity free asset can be relaxed; however, it will put more restrictions on the underlying utility function, like time separability and analyticity at 0.

Chapter 4 tests the restrictions of smooth ambiguity preference with a particular shape (decreasing or increasing ambiguity aversion) on portfolio choice, and gives lower and upper bounds for these measures. Recent papers in economics and finance show that the implication of the smooth ambiguity model crucially depends on the shape of the risk and ambiguity aversion and their magnitudes. Varian (1988) derives the nonparametric restrictions of decreasing or increasing absolute (increasing) risk aversion on one observation of Arrow−Debreu security, and gives the lower and upper bounds for these measures. I revisit Varian's argument, and extend it to multiple observations in incomplete markets. The conditions involve the existence of Afriat numbers, and can be used to bound risk aversion. Then the restrictions of different shapes of ambiguity aversion are derived.

Chapter 5 investigates systematically the nature of household ambiguity preferences using household survey data within a parametric framework. Despite the importance of individual risk aversion and ambiguity aversion in determining individual decision making and equilibrium implications, there is rare evidence on the shape of individual ambiguity preferences, except a little evidence from either lab experiments or pure thought experiments using variants of Ellsberg's urns. I derive an explicit solution in a two-period smooth ambiguity model due to Klibanoff, Marinacci, and Mukerji (2005) assuming constant relative risk aversion and constant relative ambiguity aversion, and show that time preference, risk aversion and ambiguity aversion can be uniquely identified from a special panel dataset. From the Bank of Italy Survey on Household Income and Wealth (SHIW) 2008 and 2010, the most important findings are: constant relative risk aversion and relative ambiguity aversion can be a good approximation; the recovered preference parameters display considerable heterogeneity; the average relative risk aversion is much smaller than 1; and the average relative ambiguity aversion is around 3 or larger. Other interesting findings include, firstly, households' expectations are very pessimistic, and are subject to much ambiguity; secondly, the over-identification restriction implied by the subjective expected utility model rejects the null hypothesis that households are subjective expected utility maximizers, in favor of the ambiguity model; finally, household risk aversion and ambiguity aversion are not correlated, can't be explained by observable household characteristics, and have a quantitatively significant effect on consumption and portfolio holding.

This dissertation contributes to understanding smooth ambiguity preference both theoretically and empirically. Chapters 2-4 provide theoretical foundation for future empirical work on testing and recovering individual smooth ambiguity preference. Chapter 5 provides fresh empirical evidence on the shape and magnitude of household risk and ambiguity aversion based on household consumption and saving data, which sheds light on the distribution of ambiguity preference, and can be used in finance and macroeconomic calibration to examine the implication of ambiguity preference models.

In the next section of this chapter, I set up the economic environment in which a consumer makes his choice. This model setup and notations will be used in the following chapters, and will not be repeated.

1.2 Setup

In this dissertation, I consider a one-good two-period economy. Assume states of the world in the second period are represented by $\omega \in \boldsymbol{\Omega}$, a finite set that has a product structure: $\boldsymbol{\Omega} = \boldsymbol{A} \times \boldsymbol{S}$, where $a \in \boldsymbol{A}$ are states of uncertainty while $s \in \boldsymbol{S}$ are states of risk.[1] $\boldsymbol{\Omega}$ can be interpreted as the set of possible outcomes of two-stage lotteries, where $\boldsymbol{A}$ includes outcomes of the first stage lotteries, and $\boldsymbol{S}$ includes realizations of the second stage lotteries. I follow the literature, and assume that the consumption and the payoff of financial assets are contingent on the realization of risk states only.[2] For a finite set $\boldsymbol{E}$, let $\boldsymbol{\Delta E}$ be the set of probability measures on $\boldsymbol{E}$, or, equivalently, the simplex of dimension $(\#\boldsymbol{E}-1)$. A probability measure on the set of the states of the world is $\boldsymbol{\pi} \in \boldsymbol{\Delta}(\boldsymbol{\Omega})$, and it decomposes

$$\boldsymbol{\pi} = (\boldsymbol{\mu} \otimes \boldsymbol{\nu})$$

into a probability measure over states of the uncertainty

$$\boldsymbol{\mu} \in \boldsymbol{\Delta}(\boldsymbol{A})$$

, and a family of conditional probability measures over states of risk

$$\boldsymbol{\nu}_a : \boldsymbol{A} \to \boldsymbol{\Delta}(\boldsymbol{S});$$

evidently,

$$\pi(a, s) = \mu(a)\nu(s|a).$$

From now on, for notational ease, I denote $\pi(a, s)$ by π_{as}, $\mu(a)$ by μ_a, and $\nu(s|a)$ by ν_{as}.

A distribution of wealth across the risk states is

$$\boldsymbol{x} = (..., x_s, ...) \in \times_s (0, \infty).$$

[1] The exception is Chapter 5, where I assume lognormal distribution with a continuum of states.

[2] If consumption and asset payoff depend on both uncertainty and risk states, as can happen when the two-stage lottery is a compound *objective* lottery, my arguments survive with a bit modification of notation.

A utility function over distribution of wealth is

$$U : \times_s (0, \infty) \to \mathbb{R}.$$

It is continuous, strictly monotonically increasing and strictly quasi−concave. In the interior of its domain of definition, $\boldsymbol{D}U \gg \boldsymbol{0}$, and $\boldsymbol{D}^2 U$ is negative definite on the orthogonal complement of the gradient, $\boldsymbol{D}U^{\perp}$.

Savage (1954) or, alternatively, Anscombe and Aumann (1963) derive a probability measure

$$\boldsymbol{\pi} = \boldsymbol{\mu} \otimes \boldsymbol{\nu}$$

, and a cardinal risk−uncertainty index

$$u : (0, \infty) \to \mathbb{R},$$

such that

$$U(\boldsymbol{x}) = E_{\boldsymbol{\pi}} u(x_s) = E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} u(x_s).^{3} \tag{1.1}$$

Gilboa and Schmeidler (1989) derive a convex set of probability measures

$$\boldsymbol{C} \subset \boldsymbol{\Delta}(\boldsymbol{\Omega})$$

and a risk index

$$u : (0, \infty) \to \mathbb{R},$$

such that

$$U(\boldsymbol{x}) = \min_{\boldsymbol{\pi} \in \boldsymbol{C}} E_{\boldsymbol{\pi}} u(x_s). \tag{1.2}$$

Klibanoff, Marinacci, and Mukerji (2005) derive a probability measure

$$\boldsymbol{\pi} = \boldsymbol{\mu} \otimes \boldsymbol{\nu},$$

and a (cardinal) risk index and an uncertainty index

$$u : (0, \infty) \to \mathbb{R} \quad \text{and} \quad \tilde{\phi} : u((0, \infty)) \to \mathbb{R},$$

[3] In the original formulation in Savage (1954), the domain of preference does not include compound lotteries; however, it is shown by Anscombe and Aumann (1963), and Segal (1990) that the reduction of compound lotteries is necessary for the expected utility theory when the domain includes two-stage lotteries.

respectively, such that

$$U(\boldsymbol{x}) = E_{\boldsymbol{\mu}}\tilde{\phi}\big(E_{\boldsymbol{\nu}_a}u(x_s)\big).$$

Alternatively,

$$\phi : (0, \infty) \to \mathbb{R}$$

and

$$U(\boldsymbol{x}) = E_{\boldsymbol{\mu}}\phi\Big(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(x_s)\big)\Big), \tag{1.3}$$

which, as in Selden and Wei (2014), also Hayashi and Miao (2011), allows for invariance to an increasing affine transformation of the risk index and generate appropriate comparative statics. The individual is ambiguity averse if ϕ is a concave transformation of u, and ambiguity neutral if $\phi \circ u^{-1}$ is linear. Evidently, $\phi = \tilde{\phi} \circ u$ establishes the equivalence of the formulations.

Within the framework of Anscombe and Aumann (1963), $\boldsymbol{\mu}$ is the subjective probability over horse race lotteries, and $\boldsymbol{\nu}$ is the objective probability over roulette wheels. As shown by experimental evidence in Halevy (2007), the non-reduction of compound *objective* lotteries is also consistent with the smooth ambiguity aversion, in which case both $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are objective probabilities.

More generally, individual ambiguity preference can be defined without reference to the probability of ambiguity states:

$$U(\boldsymbol{x}) = \Phi\big(w(\boldsymbol{x})\big) = \Phi\big(..., w_a(\boldsymbol{x}), ...\big), \tag{1.4}$$

where

$$w_a(\boldsymbol{x}) = u^{-1}(E_{\boldsymbol{\nu}_a}u(x_s)),$$

is the certainty equivalent wealth at a state of uncertainty, a;

$$\boldsymbol{\nu} : \boldsymbol{A} \to \boldsymbol{\Delta}(\boldsymbol{S}),$$

is a family of conditional probability measures over states of risk;

$$w(\boldsymbol{x}) = (..., w_a(\boldsymbol{x}), ...),$$

the distribution of certainty equivalent wealth across states of uncertainty;

$$u : (0, \infty) \to \mathbb{R}$$

is a risk index; and

$$\Phi : \times_a (0, \infty) \to \mathbb{R}$$

is an ordinal utility function over the distribution of certainty equivalent wealth across states of uncertainty.

Evidently, others are special cases. And, there is no explicit reference to a probability measure over states of uncertainty.

There are J assets, where J is a finite number. Payoffs of asset j are

$$\boldsymbol{r}_j = (..., r_{sj}, ...)',$$

a column vector with S rows, i.e. its payoff is risk state contingent. Without loss of generality, let

$$E_{\boldsymbol{\pi}} \boldsymbol{r}_j = 1.$$

At a state of risk, payoffs of assets are

$$\boldsymbol{R}_s = (..., r_{sj}, ...),$$

a row vector with J columns. The matrix of asset payoffs is

$$\boldsymbol{R} = (..., \boldsymbol{r}_j, ...) = (..., \boldsymbol{R}_s, ...),$$

a matrix of full column rank.

A portfolio of assets is

$$\boldsymbol{y} = (..., y_j, ...).$$

It generates the distribution of wealth across states of risk

$$\boldsymbol{x} = \boldsymbol{R}\boldsymbol{y}.$$

The set of portfolios of assets that generate strictly positive distributions of wealth across states of risk is non-empty,

$$\boldsymbol{Y} = \{\boldsymbol{y} : \boldsymbol{R}\boldsymbol{y} \gg \mathbf{0}\} \neq \emptyset;$$

since $\boldsymbol{R}$ is of full column rank, $\boldsymbol{Y}$ is open. The domain of prices of assets that do not allow for arbitrage is

$$\boldsymbol{P} = \{\boldsymbol{p} : \boldsymbol{R}\boldsymbol{y} > 0 \Rightarrow \boldsymbol{p}\boldsymbol{y} > 0\} = \{\boldsymbol{p} = \boldsymbol{\Pi}\boldsymbol{R}, \boldsymbol{\Pi} \gg \mathbf{0}\}.$$

Chapter 2

Testing smooth ambiguity preference

2.1 Introduction

The standard consumer theory assumes that a rational consumer will maximize his well-behaved utility function subject to a budget constraint. In certainty case, the testable implications of such theory have been derived from two different approaches. The first approach dated back to Antonelli (1971) and Slutsky (1960) is to assume the whole demand function is observable, and derives the necessary and sufficient condition on the derivative of demand functions, which is known as Slutsky equation.[1] The second approach attributed to Samuelson (1938), Afriat (1967), Diewert (1973) and Varian (1982) is to assume finite data observation, and derive the necessary and sufficient condition to be compatible with utility maximization, which is known as Afriat theorem, see Kreps (2013).[2]

Under risk, a rational consumer's preference is expressed as expected utility in von Neumann and Morgenstern (1944), Savage (1954) and Anscombe and Aumann (1963). The testable restrictions of expected utility theory

[1]For the Slutsky condition to be sufficient, income-Lipschitz condition of the demand functions-i.e. the derivative of the demand functions with respect to income is bounded on the domain, is needed, see Hurwicz and Uzawa (1971).

[2]This approach is called revealed preference approach since originally the conditions are expressed as Weak Axiom of Reveal Preference by Samuelson (1938). Afriat (1967) charaterize the restrictions by a system of inequalities called Afriat inequalities. Varian (1982) proves that the Afriat inequalities are equivalent to Generalized Axiom of Revealed Preference.

have been exploited using both approaches. Kubler, Selden, and Wei (2014) derive Slutsky conditions under complete markets, which involves derivatives of demand function with respect to probability. Basically, they work in the framework of von Neumann and Morgenstern (1944), and they assume probability is observable and changes across observations. They also give a functional form test, where the contingent claim demand must satisfy certain form restrictions. Polemarchakis and Selden (1983) derive the Slutsky-like conditions under incomplete markets, where the probability is fixed, and the conditions involve the existence of some unknown functions. The revealed preference conditions have been developed by Green and Srivastava (1986), and Kubler, Selden, and Wei (2014). The model setup in Green and Srivastava (1986) is quite general, and is applicable to incomplete markets, and multiple goods in each state. The necessary and sufficient conditions there involve existence of unknown utility levels and multipliers. Kubler, Selden, and Wei (2014) assume one good in each state under complete markets, and characterize the necessary and sufficient conditions by strong axiom of revealed expected utility (SAREU), which is free of existential quantifiers.

The expected utility theory has been challenged by Ellsberg's paradox Ellsberg (1961) and other experimental evidence, which demonstrate that individual's choice will violate the independence axiom when he is ambiguous about the probability distribution of relevant events.[3] In recent years, the decision theory literature has developed alternative models to accommodate individual choice behavior under ambiguity. In developing laboratory tests of whether a particular model provides a satisfactory description of choices of individuals, decision theorists have focused on choices over lotteries whereas the economists alternatively have focused on choices over assets (or contingent claims). One potentially important limitation of the former is that the choices over lotteries reflect neither variable prices for lotteries nor budget constraints associated with a fixed income. In recent work that overcomes the latter problem, there has been considerable focus on developing revealed preference tests, where following the classic work of Afriat (1967) and Varian (1982), one seeks to verify whether observed asset (price,quantity) pairs are consistent with specific demand tests derived for

[3]For the early evidence, see the survey article Camerer and Weber (1992).

assumed non-parametric forms of utility (such as additive separability or weak separability). Two important applications of this approach to ambiguity preferences are Bayer, Bose, Polisson, and Renou (2013) and Ahn, Choi, Gale, and Kariv (2014). For instance in the former, the authors derive testable inequality conditions, associated with the assumed risk and ambiguity indices, which are consistent with maximization of ambiguity preferences. This work, while extremely interesting, would seem to have two limitations. First, only very special non-parametric forms of ambiguity preferences can be addressed. Second, almost all of the revealed preference work in risky or uncertain settings of which we are aware makes the very strong assumption of complete asset markets where the number of states of nature equals the number of assets, which span the state space.

In this chapter, we use both revealed preference approach and demand function approach to test smooth ambiguity model. In both approaches, we assume the probability distributions are known, and change across observations.[4] The revealed preference approach does not put any requirement on the financial market, i.e. the necessary and sufficient conditions can be used to test portfolio choice from incomplete markets. For demand function tests, we assume complete state consumption contingent on both ambiguity and risk states.

2.2 Revealed preference test

As mentioned in Section 1.2 in Chapter 1, the ambiguity preference can be written more generally as

$$U(\boldsymbol{x}) = \Phi\Big(..., E_{\boldsymbol{\nu}_a} u(x_a), ...\Big). \tag{2.1}$$

We will put some regularity assumptions on the functional form (2.1) and asset return structure to make the individual optimization problem well-defined.

Assumption 1

(1) u is $\boldsymbol{C}^2$ on $\mathbb{R}_{++}$, is strictly concave and satisfies $\forall x \in \mathbb{R}_{++}$, $u^{'} > 0$;

[4]As will be seen, observing the probability distribution is not necessary for the revealed preference test.

(2) Φ is $\boldsymbol{C}^1$ on $\mathbb{R}^A_{++}$, is strictly concave, and satisfies $\forall \boldsymbol{w}(\boldsymbol{x}) \in \mathbb{R}^A_{++}$, $\Phi_a > 0$ for all $a = 1, ..., A$;

(3) $\Phi\big(..., u^{-1}(E_{\boldsymbol{\nu}_a} u(x_s)), ...\big)$ is strictly quasi-concave on $\mathbb{R}^S_{++}$.

Assumption 2 For each conditional probability distribution $\boldsymbol{v}_a$, the gross return $\boldsymbol{r}_j$ of asset j, $j = 1, ..., J$ satisfies:

(1) $prob\{\boldsymbol{r}_j \geq 0\}$ =1;

(2) $prob\{\boldsymbol{r}_j = 0\} \neq 1$;

(3) $\boldsymbol{r}_j$ is linearly independent with return vectors of other assets;

(4) $E_{\boldsymbol{\nu}_a} \boldsymbol{r}^l_j < +\infty$.

Now we specify the data we can observe: data $\boldsymbol{\mathfrak{D}}$=$\{\boldsymbol{p}^n, \boldsymbol{y}^n, \boldsymbol{\nu}^n_a, \boldsymbol{R}\}^{n=1,...,N}_{a=1,...,A}$. So the data $\boldsymbol{\mathfrak{D}}$ include N observations of asset prices $\boldsymbol{p}$, portfolio choices $\boldsymbol{y}$, all conditional probability distributions $\boldsymbol{v}$, and the asset payoff structure $\boldsymbol{R}$, which satisfies Assumption 2.

Remark 1. No matter the domain of preference is subjective-objective two-stage lotteries as in Anscombe and Aumann (1963) or a compound *objective* lotteries as shown by Halevy (2007), the conditional distributions are objective. So assuming observation of conditional probability distribution seems to be a tenable assumption. In the experiments of Ellsberg (1961) or other experiments like Ahn, Choi, Gale, and Kariv (2014), the conditional probabilities are objectively known to the subjects.

To test the ambiguity preference (2.1) based on finite observations in dataset $\boldsymbol{\mathfrak{D}}$, we extend the revealed preference method developed by Afriat (1967), Varian (1982) and Matzkin and Richter (1991) to our setting. Proposition 1 presents the necessary and sufficient conditions for finite observations in dataset $\boldsymbol{\mathfrak{D}}$ to be consistent with such ambiguity preference.

Proposition 1. *The following conditions are equivalent:*

(i) *There exists a continuous, locally non-satiated utility function*

$$U(\boldsymbol{y}) = \Phi\left(..., \sum_{s=1}^{S} \nu_{as} u\left(\sum_{j=1}^{J} r_{sj} y_j\right), ...\right),$$

where Φ *and* u *satisfy Assumption 1, to rationalize the data* $\mathfrak{D}$, *i.e. for all* $n = 1, ..., N$

$$\boldsymbol{y}^n \in \arg\max_{\boldsymbol{y} \in \mathbb{R}^J} U(\boldsymbol{y}) \quad s.t.\ \boldsymbol{p}^n \cdot \boldsymbol{y} \leq \boldsymbol{p}^n \cdot \boldsymbol{y}^n.$$

(ii) *There exist real numbers* $(U_s^n, M_s^n)_{s=1,...,S}^{n=1,...,N} > 0$, $(\Phi^n)_{n=1}^N$, $(\rho_a^n)_{a=1,...,A}^{n=1,...,N} > 0$, *and* $(\lambda^n)_{n=1}^N > 0$ *such that for all* $n, m \in \{1, 2, ..., N\}$, $s, s' \in \{1, 2, ..., S\}$, $a, a' \in \{1, 2, ..., A\}$ *and* $j \in \{j = 1, 2, ..., J\}$ [5]

$$U_s^n - U_{s'}^m < M_{s'}^m \left(\sum_{j=1}^J r_{sj} y_j^n - \sum_{j=1}^J r_{s'j} y_j^m \right),$$

with equality if $\sum_{j=1}^J r_{sj} y_j^n = \sum_{j=1}^J r_{s'j} y_j^m$;

$$\Phi^n - \Phi^m < \sum_{a=1}^A \rho_a^m \left(\sum_{s=1}^S \nu_{as}^n U_s^n - \sum_{s=1}^S \nu_{a's}^m U_s^m \right),$$

with equality if $\sum_{s=1}^S \nu_{as}^n U_s^n = \sum_{s=1}^S \nu_{a's}^m U_s^m$;

and

$$\sum_{a=1}^A \left(\rho_a^n \sum_{s=1}^S \nu_{as}^n M_s^n r_{sj} \right) = \lambda^n p_j^n.$$

Proof. (i) implies (ii)

The first order conditions are $\forall j \in \{1, 2, ..., J\}$,

$$\sum_{\mathbf{a=1}}^A \Phi_a^{'} \left(..., \sum_{s=1}^S \nu_{as}^n u \left(\sum_{j=1}^J r_{sj} y_j^n \right), ... \right) \left(\sum_s \nu_{as}^n u' \left(\sum_{j=1}^J r_{sj} y_j^n \right) r_{sj} \right) = \lambda p_j^n. \tag{2.2}$$

Since u and Φ are both strictly concave, the following inequalities hold

$$u \left(\sum_{j=1}^J r_{sj} y_j^n \right) < u \left(\sum_{j=1}^J r_{s'j} y_j^m \right) + u' \left(\sum_{j=1}^J r_{s'j} y_j^m \right) \left(\sum_{j=1}^J r_{sj} y_j^n - \sum_{j=1}^J r_{s'j} y_j^m \right), \tag{2.3}$$

[5] In this condition, the numbers $(U_s^n)_{s=1,...,S}^{n=1,...,N}$ represent the utility levels, which are not necessarily positive. However, the translation of any negative solution by a positive constant will still be a solution. So requiring positivity of these numbers is without loss of generality.

and

$$
\begin{aligned}
&\Phi\left(\dots,\sum_{s=1}^{S}\nu_{as}^{n}u\left(\sum_{j=1}^{J}r_{sj}y_{j}^{n}\right),\dots\right)\\
&<\Phi\left(\dots,\sum_{s=1}^{S}\nu_{as}^{m}u\left(\sum_{j=1}^{J}r_{sj}y_{j}^{m}\right),\dots\right)+\sum_{a=1}^{A}\Phi'\left(\dots,\sum_{s=1}^{S}\nu_{as}^{m}u\left(\sum_{j=1}^{J}r_{sj}y_{j}^{m}\right),\dots\right)\\
&\times\left(\sum_{s=1}^{S}\nu_{as}^{n}u\left(\sum_{j=1}^{J}r_{sj}y_{j}^{n}\right)-\sum_{s=1}^{S}\nu_{a's}^{m}u\left(\sum_{j=1}^{J}r_{sj}y_{j}^{m}\right)\right).
\end{aligned}
\tag{2.4}
$$

Denoting

$$
U_s^n=u\left(\sum_{j=1}^{J}r_{sj}y_j^n\right),\ M_s^n=u'\left(\sum_{j=1}^{J}r_{sj}y_j^n\right),
$$

$$
\Phi^n=\Phi\left(\dots,\sum_{s=1}^{S}\nu_{as}^{n}u\left(\sum_{j=1}^{J}r_{sj}y_{j}^{n}\right),\dots\right),
$$

$$
\rho_a^n=\Phi_a'\left(\dots,\sum_{s=1}^{S}\nu_{as}^{n}u\left(\sum_{j=1}^{J}r_{sj}y_{j}^{n}\right),\dots\right).
$$

The conditions (2.2), (2.3), and (2.4) can be rewritten as

$$
U_s^n-U_{s'}^m<M_{s'}^m\left(\sum_{j=1}^{J}r_{sj}y_j^n-\sum_{j=1}^{J}r_{s'j}y_j^m\right),
\tag{2.5}
$$

$$
\Phi^n-\Phi^m<\sum_{a=1}^{A}\rho_a^m\left(\sum_{s=1}^{S}\nu_{as}^nU_s^n-\sum_{s=1}^{S}\nu_{a's}^mU_s^m\right),
\tag{2.6}
$$

and

$$
\sum_{a=1}^{A}\left(\rho_a^n\sum_{s=1}^{S}\nu_{as}^nM_s^nr_{sj}\right)=\lambda^np_j^n.
\tag{2.7}
$$

(ii) implies (i)

Given the solution to inequalities in (ii), i.e. real numbers $(U_s^n, M_s^n)_{s=1,\dots,S}^{n=1,\dots,N} > 0$, $(\Phi^n)_{n=1}^N$, $(\rho_a^n)_{a=1,\dots,A}^{n=1,\dots,N} > 0$ and $(\lambda^n)_{n=1}^N > 0$, we will modify the argument in Matzkin and Richter (1991) to construct strictly increasing and strictly concave utility indexes $u(x)$ and $\Phi(\dots,u_a,\dots)$ to rationalize the observations.

Step 1: construction of $u(x)$

Since we have only finite inequalities, we can choose small enough number δ_0 such that

$$U_s^n - U_{s'}^m < M_{s'}^m \left(\sum_{j=1}^{J} r_{sj} y_j^n - \sum_{j=1}^{J} r_{s'j} y_j^m \right) - \delta_0, \tag{2.8}$$

for $\sum_{j=1}^{J} r_{sj} y_j^n \neq \sum_{j=1}^{J} r_{s'j} y_j^m$.

Define a function $g : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$g(x) = (x^2 + T)^{\frac{1}{2}} - T^{\frac{1}{2}}, \tag{2.9}$$

where T is a positive real number.

It can be shown that the defined function $g(x)$ is nonnegative valued, differentiable, strictly convex, and has bounded derivative. And condition (2.8) implies that we can choose a small enough number δ such that

$$U_s^n - U_{s'}^m < M_{s'}^m \left(\sum_{j=1}^{J} r_{sj} y_j^n - \sum_{j=1}^{J} r_{s'j} y_j^m \right) - \delta g \left(\sum_{j=1}^{J} r_{sj} y_j^n - \sum_{j=1}^{J} r_{s'j} y_j^m \right), \tag{2.10}$$

for $\sum_{j=1}^{J} r_{sj} y_j^n \neq \sum_{j=1}^{J} r_{s'j} y_j^m$.

Define functions $u_s^n(x) : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$u_s^n(x) = U_s^n + M_s^n \left(x - \sum_{j=1}^{J} r_{sj} y_j^n \right) - \delta g \left(x - \sum_{j=1}^{J} r_{sj} y_j^n \right), \tag{2.11}$$

where $n \in \{1, 2, ..., N\}$, $s \in \{1, 2, ..., S\}$.

It can be shown that the defined function $u_s^n(x)$ is strictly concave, and satisfies $u_s^n \left(\sum_{j=1}^{J} r_{sj} y_j^n \right) = U_s^n$.

Define a function $u(x) : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$u(x) = \min_{s,n} \{u_s^n(x)\}. \tag{2.12}$$

The defined function $u(x)$ is strictly concave, and we can choose δ small enough that the function $u(x)$ is strictly increasing. This is possible since the defined function $g(x)$ has bounded derivative and there are finite inequalities.

We claim that $u(\sum_{j=1}^{J} r_{sj} y_j^n) = U_s^n$, since

$$\begin{aligned} u\left(\sum_{j=1}^{J} r_{sj} y_j^n\right) &= u_{s'}^m \left(\sum_{j=1}^{J} r_{sj} y_j^n\right) \\ &\leq u_s^n \left(\sum_{j=1}^{J} r_{sj} y_j^n\right) \\ &= U_s^n \end{aligned} \tag{2.13}$$

The inequality in above equation can not be strict, otherwise it will violate inequality (2.10).

Step 2: construction of index $\Phi(\boldsymbol{u})$:

We will sketch the construction, since it follows the same argument as above.

Define a function $G(\boldsymbol{u}) : \mathbb{R}^A \to \mathbb{R}^1$ by

$$G(\boldsymbol{u}) = (u_1^2 + ...u_A^2 + T)^{\frac{1}{2}} - T^{\frac{1}{2}}. \tag{2.14}$$

Choose small enough positive real number ϵ such that

$$\begin{aligned} \Phi^n - \Phi^m < &\sum_{a=1}^{A} \rho_a^m \left(\sum_{s=1}^{S} \nu_{as}^n U_s^n - \sum_{s=1}^{S} \nu_{a's}^m U_s^m\right) \\ &- \epsilon G\left(..., \sum_{s=1}^{S} \nu_{as}^n U_s^n - \sum_{s=1}^{S} \nu_{a's}^m U_s^m, ...\right), \end{aligned} \tag{2.15}$$

for $\sum_{s=1}^{S} \nu_{as}^n U_s^n \neq \sum_{s=1}^{S} \nu_{a's}^m U_s^m$.

Define functions $\phi^n(\boldsymbol{u}) : \mathbb{R}^A \to \mathbb{R}^1$ by

$$\phi^n(\boldsymbol{u}) = \Phi^n + \sum_{a=1}^{A} \rho_a^m \left(u_a - \sum_{s=1}^{S} \nu_{a's}^m U_s^m \right) - \epsilon G \left(..., u_a - \sum_{s=1}^{S} \nu_{a's}^m U_s^m, ... \right), \tag{2.16}$$

where $n \in \{1, 2, ..., N\}$, $a \in \{1, 2, ..., A\}$.

It can be shown that the defined function $\phi^n(\boldsymbol{u})$ is strictly concave and satisfies $\phi^n \left(..., \sum_{s=1}^{S} \nu_{as}^n U_s^n, ... \right) = \Phi^n$.

Define a function $\Phi : \mathbb{R}^A \to \mathbb{R}^1$ by

$$\Phi(\boldsymbol{u}) = \min_n \{\phi^n(\boldsymbol{u})\}. \tag{2.17}$$

The defined function $\Phi(\boldsymbol{u})$ is strictly concave, and we can choose ϵ small enough such that $\Phi(\boldsymbol{u})$ is strictly increasing. It can be shown that $\Phi \left(..., \sum_{s=1}^{S} \nu_{as}^n U_s^n, ... \right) = \Phi^n$.

Step 3: rationalization

We claim that the constructed utility function rationalizes the observed data, i.e. if $\boldsymbol{p}^i \boldsymbol{y}^i > \boldsymbol{p}^i \boldsymbol{y}$ and $\boldsymbol{y}^i \neq \boldsymbol{y}$, then

$$\Phi \left(..., \sum_{s=1}^{S} \nu_{as}^i u \left(\sum_{j=1}^{J} r_{sj} y_j^i \right), ... \right) > \Phi \left(..., \sum_{s=1}^{S} \nu_{as}^i u \left(\sum_{j=1}^{J} r_{sj} y_j \right), ... \right). \tag{2.18}$$

$$
\begin{aligned}
&\Phi\left(\ldots,\sum_{s=1}^{S}\nu_{as}^{i}u\left(\sum_{j=1}^{J}r_{sj}y_j\right),\ldots\right)\\
&\overset{①}{=}\min_m\Big\{\Phi^m+\sum_{a=1}^{A}\rho_a^m\Big(\sum_{s=1}^{S}\nu_{as}^{i}u\big(\sum_{j=1}^{J}r_{sj}y_j\big)-\sum_{s}^{S}\nu_{a's}^{m}U_s^m\Big)\\
&-\epsilon G\Big(\ldots,\sum_{s=1}^{S}\nu_{as}^{i}u\big(\sum_{j=1}^{J}r_{sj}y_j\big)-\sum_{s}^{S}\nu_{a's}U_s^m,\ldots\Big)\Big\}\\
&\overset{②}{=}\min_m\Big\{\Phi^m+\sum_{a=1}^{A}\rho_a^m\Big(\sum_{s=1}^{S}\nu_{as}^{i}\min_{s',n}\big\{U_{s'}^n+M_{s'}^n\big(\sum_{j=1}^{J}r_{sj}y_j-\sum_{j=1}^{J}r_{s'j}y_j^n\big)\\
&-\delta g\big(\sum_{j=1}^{J}r_{sj}y_j-\sum_{j=1}^{J}r_{s'j}y_j^n\big)\big\}-\sum_{s}^{S}\nu_{a's}^{m}U_s^m\Big)\\
&-\epsilon G\Big(\ldots,\sum_{s=1}^{S}\nu_{as}^{i}u\big(\sum_{j=1}^{J}r_{sj}y_j\big)-\sum_{s}^{S}\nu_{a's}^{m}U_s^m,\ldots\Big)\Big\}\\
&\overset{③}{\leq}\Phi^i+\sum_{a=1}^{A}\rho_a^i\Big(\sum_{s=1}^{S}\nu_{as}^{i}U_s^i+M_s^i\big(\sum_{j=1}^{J}r_{sj}y_j-\sum_{j=1}^{J}r_{sj}y_j^i\big)\\
&-\delta g\big(\sum_{j=1}^{J}r_{sj}y_j-\sum_{j=1}^{J}r_{sj}y_j^i\big)-\sum_{s}^{S}\nu_{as}U_s^i\Big)\\
&-\epsilon G\Big(\ldots,\sum_{s=1}^{S}\nu_{as}u\big(\sum_{j=1}^{J}r_{sj}y_j\big)-\sum_{s}^{S}\nu_{as}^{i}U_s^i,\ldots\Big)\\
&\overset{④}{<}\Phi^i+\sum_{a=1}^{A}\rho_a^i\Big(\sum_{s=1}^{S}\nu_{as}^{i}U_s^i+M_s^i\big(\sum_{j=1}^{J}r_{sj}y_j-\sum_{j=1}^{J}r_{sj}y_j^i\big)-\sum_{s}^{S}\nu_{as}^{i}U_s^i\Big)\\
&=\Phi^i+\sum_{a=1}^{A}\rho_a^i\sum_{s=1}^{S}\nu_{as}^{i}M_s^i\Big(\sum_{j=1}^{J}r_{sj}y_j-\sum_{j=1}^{J}r_{sj}y_j^i\Big)\\
&\overset{⑤}{=}\Phi^i+\lambda^i p^i(y-y^i)\\
&\overset{⑥}{\leq}\Phi^i\\
&\overset{⑦}{=}\Phi\Big(\ldots,\sum_{s=1}^{S}\nu_{as}^{i}U_s^i,\ldots\Big)\\
&\overset{⑧}{=}\Phi\Big(\ldots,\sum_{s=1}^{S}\nu_{as}^{i}u\big(\sum_{j=1}^{J}r_{sj}y_j^i\big),\ldots\Big).
\end{aligned}
$$

where ① follows from the definition of function Φ, ② from the definition of function u, ③ from taking the minimum, ④ from positivity of functions g and G, ⑤ from equation (2.7), and ⑥ from the budget constraint. $\square$

Remark 2. As in Matzkin and Richter (1991), we can prove the constructed function is generically infinitely differentiable. And if we put further restrictions: $M_s^n = M_{s'}^m$ if $\sum_{j=1}^J r_{sj} y_j^n = \sum_{j=1}^J r_{s'j} y_j^m$, and $\rho_a^n = \rho_{a'}^m$ if $\sum_{s=1}^S \nu_{as}^n U_s^n = \sum_{s=1}^S \nu_{a's}^m U_s^m$, then we can use the convolution methods in Chiappori and Rochet (1987) to smooth our defined functions to be infinitely differentiable on the whole domain. So with finite observations, differentiability is not testable, in the first statement the differentiability of the objective function is not needed.

Remark 3. In the above testing, we assume the conditional probability is observed, and varies across observations. The assumption of observing conditional probability can be relaxed, in stead, we can require the existence of these numbers in condition (ii); however, in this case, we need to assume these probabilities are fixed across observations, otherwise the testability will be lost.

The smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005) is a special case when

$$\Phi(\ldots, E_{\boldsymbol{\nu}_a} u(x_s), \ldots) = \sum_{a=1}^{A} \mu_a \phi \left(\sum_{s=1}^{S} \nu_{as} u\left(x_s\right) \right). \tag{2.19}$$

Here the probability of ambiguity states is explicitly referred to. We put the following restrictions on the functional form (2.19).

Assumption $1^{'}$

(1) u is $\boldsymbol{C}^2$ on $\mathbb{R}_{++}$, is strictly concave and satisfies $\forall x \in \mathbb{R}_{++}$, $u^{'} > 0$;

(2) ϕ is $\boldsymbol{C}^2$ on $\mathbb{R}_{++}$, is strictly concave, and satisfies $\forall w(\boldsymbol{x}) \in \mathbb{R}_{++}$, $\phi^{'} > 0$;

(3) $\phi\big(u^{-1}(.)\big)$ is strictly concave on $\mathbb{R}$.

To test this functional form, we specify the following data set.

Data $\boldsymbol{\mathfrak{D}'}$=$\{\boldsymbol{p}^n, \boldsymbol{y}^n, \boldsymbol{\mu}^n, \boldsymbol{\nu}_a^n, \boldsymbol{R}\}_{a=1,\ldots,A}^{n=1,\ldots,N}$.

So the data $\mathfrak{D}'$ includes N observations of asset prices $\boldsymbol{p}$, portfolio choices $\boldsymbol{y}$, the probability distribution over ambiguity states $\boldsymbol{\mu}$, all conditional probability distributions $\boldsymbol{v}$, and the asset payoff structure $\boldsymbol{R}$, which satisfies Assumption 2.

Remark 4. We assume the observation of both $\boldsymbol{\mu}$, the probability over ambiguity states, and $\boldsymbol{\nu}$, the conditional probability over risk states. If the domain of preference is compound *objective* lotteries, such assumption seems to be reasonable. Within the framework of Anscombe and Aumann (1963), it seems much less plausible to observe probability $\boldsymbol{\mu}$. In the following corollary, we state the necessary and sufficient conditions assuming observation of $\boldsymbol{\mu}$; however, as pointed out in remark 3, we can require the existence of these numbers when they are not observable.

Corollary 1 gives the corresponding necessary and sufficient conditions for data $\mathfrak{D}'$ to be consistent with this particular functional form.

Corollary 1. *The following conditions are equivalent:*

(i) *There exists a continuous, locally non-satiated utility function*

$$U(\boldsymbol{y}) = \sum_{a=1}^{A} \mu_a \phi \left(\sum_{s=1}^{S} \nu_{as} u \left(\sum_{j=1}^{J} r_{sj} y_j \right) \right),$$

where ϕ and u satisfy Assumption $1'$, to rationalize the data $\mathfrak{D}'$, i.e. for all $i = 1, ..., N$

$$\boldsymbol{y}^n \in \arg\max_{\boldsymbol{y} \in \mathbb{R}^J} U(\boldsymbol{y}) \quad s.t.\ \boldsymbol{p}^n \cdot \boldsymbol{y} \leq \boldsymbol{p}^n \cdot \boldsymbol{y}^n.$$

(ii) *There exist real numbers $(U_s^n, M_s^n)_{s=1,...,S}^{n=1,...,N} > 0$, $(\Phi_a^n)_{a=1,...,A}^{n=1,...,N}$, $(\rho_h^n)_{a=1,...,A}^{n=1,...,N} > 0$ and $(\lambda^n)_{n=1}^N > 0$ such that for all $n, m \in \{1, 2, ..., N\}$, $s, s' \in \{1, 2, ..., S\}$, $a, a' \in \{1, 2, ..., A\}$ and $j \in \{j = 1, 2, ..., J\}$*

$$U_s^n - U_{s'}^m < M_{s'}^m \left(\sum_{j=1}^{J} r_{sj} y_j^n - \sum_{j=1}^{J} r_{s'j} y_j^m \right),$$

with equality if $\sum_{j=1}^{J} r_{sj} y_j^n = \sum_{j=1}^{J} r_{s'j} y_j^m$*;*

$$\Phi_a^n - \Phi_{a'}^m < \rho_{a'}^m \left(\sum_{s=1}^{S} \nu_{as}^n U_s^n - \sum_{s=1}^{S} \nu_{a's}^m U_s^m \right),$$

with equality if $\sum_{s=1}^{S} \nu_{as}^n U_s^n = \sum_{s=1}^{S} \nu_{a's}^m U_s^m$*; and*

$$\sum_{a=1}^{A} \left(\mu_a^n \rho_a^n \sum_{s=1}^{S} \nu_{as}^n M_s^n r_{sj} \right) = \lambda^n p_j^n.$$

Proof. We can modify the proof of Proposition 1 to prove this result. To prove (i) implies (ii), use the strict concavity of function u, the strict concavity of function ϕ, and the first order condition. To prove (ii) implies (i), the construction of $u(x)$ follows the same argument. We give a sketch of the construction of $\phi(u)$.

Construction of index $\phi(u)$*:*

Choose a small enough positive real number ϵ such that:

$$\Phi_a^n - \Phi_{a'}^m < \rho_{a'}^m \left(\sum_{s=1}^{S} \nu_{as}^n U_s^n - \sum_{s=1}^{S} \nu_{a's}^m U_s^m \right) - \epsilon g \left(\sum_{s=1}^{S} \nu_{as}^n U_s^n - \sum_{s=1}^{S} \nu_{a's}^m U_s^m \right), \tag{2.20}$$

for $\sum_{s=1}^{S} \nu_{as}^n U_s^n \neq \sum_{s=1}^{S} \nu_{a's}^m U_s^m$.

Define functions $\phi_a^n(u) : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$\phi_a^n(u) = \Phi_a^n + \rho_a^n \left(u - \sum_{s=1}^{S} \nu_{as}^n U_s^n \right) - \epsilon g \left(u - \sum_{s=1}^{S} \nu_{as}^n U_s^n \right), \tag{2.21}$$

where $n \in \{1, 2, ..., N\}$, $a \in \{1, 2, ..., A\}$.

It can be shown that the defined function $\phi_a^n(u)$ is strictly concave and satisfies $\phi_a^n \left(\sum_{s=1}^{S} \nu_{as}^n U_s^n \right) = \Phi_a^n$.

Define a function $\phi : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$\phi(u) = \min_{a,n} \{\phi_a^n(u)\}. \tag{2.22}$$

The defined function $\phi(u)$ is strictly concave, and we can choose ϵ small enough such that $\phi(u)$ is strictly increasing. It can be shown that $\phi\left(\sum_{s=1}^{S} \nu_{as}^n U_s^n\right) = \Phi_a^n$.

We omit the remaining details of rationalization. □

Remark 5. Bayer, Bose, Polisson, and Renou (2013) is a complete market version of Corollary 1, so a special case of Proposition 1, and our conditions can test observations from incomplete markets. Our conditions are stated for strictly concave functions, but if the strict inequalities are changed to be weak inequalities, they will become necessary and sufficient conditions for testing weakly concave smooth ambiguity utility under incomplete markets.

2.3 Demand function tests

In the following demand function tests, we deviate from the literature, and assume that both probability of ambiguity and probability of risk are observable, and contingent consumption can be traded contingent on both states. If the domain of preference is compound *objective* lotteries, then such assumption is tenable. As shown by experimental evidence in Halevy (2007), the non-reduction of compound *objective* lotteries is also consistent with the smooth ambiguity aversion, in which case both $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are objective probabilities. If the domain of preference is horse race-roulette wheel two-stage lotteries, the the first stage subjective probability should be elicited from subjects.

For both demand function tests, we assume the probabilities $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ change across observations. When consumers can trade consumption claims contingent on both ambiguity and risk states, their utility function is defined over distribution of consumption over states of the world:

$$U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu}) = E_{\boldsymbol{\mu}}\phi\Big(E_{\boldsymbol{\nu}_a} u(x_{as})\Big).[6] \tag{2.23}$$

[6] That probabilities $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ enter the objective function means probabilities change over observations, and it should be noted that these are exogenous to consumers, not

We are interested in consumers who are both risk and ambiguity averse, and put the following regularity restrictions on the function (2.23):

Regularity Assumption $1''$

(1) u is $\boldsymbol{C}^2$ on $\mathbb{R}_{++}$, is strictly concave and satisfies $\forall x \in \mathbb{R}_{++}$, $u' > 0$;

(2) ϕ is $\boldsymbol{C}^2$ on $\mathbb{R}$, is strictly concave, and satisfies $\forall x \in \mathbb{R}$, $\phi' > 0$.

Because both of our demand function tests crucially depend on separability of the objective function, we first give the concepts of separability we use. For n commodities, the set of these commodities is denoted by $\boldsymbol{N}$, i.e. $\boldsymbol{N} = \{1, ..., n\}$. A partition of the set $\boldsymbol{N}$ is a class of mutually exclusive and exhaustive subsets $\{\boldsymbol{N}_1, ..., \boldsymbol{N}_T\}$ such that $\boldsymbol{N} = \boldsymbol{N}_1 \cup ... \cup \boldsymbol{N}_T$, and $\boldsymbol{N}_s \cap \boldsymbol{N}_t = \boldsymbol{\emptyset}$ for $s \neq t$. So a commodity bundle $\boldsymbol{x} = (x_1, ..., x_n)$ is correspondingly partitioned into $(\boldsymbol{x}^{(1)}, ..., \boldsymbol{x}^{(T)})$, where for each t, the sub-vector $\boldsymbol{x}^{(t)}$ is composed of x_i, $i \in \boldsymbol{N}_t$.

Definition 1. A utility function $u(\boldsymbol{x})$ is *strongly separable* with respect to partition $\{\boldsymbol{N}_1, ..., \boldsymbol{N}_T\}$ with $T > 2$, if $u(\boldsymbol{x}) = u(\boldsymbol{x}^{(1)}, ..., \boldsymbol{x}^{(T)})$ is of the form

$$u(\boldsymbol{x}) = F\big(u^1(\boldsymbol{x}^{(1)}) + ... + u^T(\boldsymbol{x}^{(T)})\big), \tag{2.24}$$

where $F(y)$ is a monotone-increasing function of one variable y, and for each $t = 1, ..., T$, $u^t(\boldsymbol{x}^{(t)})$ is a function of sub-vector $\boldsymbol{x}^{(t)}$.

Definition 2. A utility function $u(\boldsymbol{x})$ is *weakly separable* with respect to partition $\{\boldsymbol{N}_1, ..., \boldsymbol{N}_T\}$ with $T > 2$, if $u(\boldsymbol{x}) = u(\boldsymbol{x}^{(1)}, ..., \boldsymbol{x}^{(T)})$ is of the form

$$u(\boldsymbol{x}) = \mathcal{F}\big(u^1(\boldsymbol{x}^{(1)}), ..., u^T(\boldsymbol{x}^{(T)})\big), \tag{2.25}$$

where $\mathcal{F}(u^1, ..., u^T)$ is a function of T variables, and for each $t = 1, ..., T$, $u^t(\boldsymbol{x}^{(t)})$ is a function of subvector $\boldsymbol{x}^{(t)}$ alone.

Remark 6. Both definitions follow Goldman and Uzawa (1964). However, in Goldman and Uzawa (1964), their primitive definition of *strong separability* is $\frac{\partial u_i(\boldsymbol{x})/u_j(\boldsymbol{x})}{\partial x_k} = 0$, for all $i \in \boldsymbol{N}_s$, $j \in \boldsymbol{N}_t$, and $k \notin \boldsymbol{N}_s \cup \boldsymbol{N}_t$ $(s \neq t)$, and that of *weak separability* is $\frac{\partial u_i(\boldsymbol{x})/u_j(\boldsymbol{x})}{\partial x_k} = 0$, for all $i, j \in \boldsymbol{N}_t$, and $k \notin \boldsymbol{N}_t$. Goldman and Uzawa (1964) prove that their primitive definitions imply the functional forms in Definition 1 and Definition 2, respectively.

choice variables.

Remark 7. The concepts of both strong and weak separability are ordinal, and a strongly separable utility function is also weakly separable.

Under the assumption of complete contingent consumption, consumer's objective function (2.23) satisfies the property of strong separability across both ambiguity states and risk states. Consumer's optimization problem is

$$\max_{\boldsymbol{x}\in\mathbb{R}^{AS}_{++}} \sum_{a=1}^{A} \mu_a \phi\left(\sum_{s=1}^{S} \nu_{as} u(x_{as})\right), \quad s.t.\ \boldsymbol{p}\cdot\boldsymbol{x} \leq I. \tag{2.26}$$

Under the regularity assumption, solution to the problem (2.26) exists and is unique. We assume the optimal consumption demands $\boldsymbol{x}(\boldsymbol{p}; I; \boldsymbol{\mu}; \boldsymbol{\nu})$ are observable. We want to know what properties these demand functions possess if they are generated from problem (2.26).

2.3.1 Functional form test

Goldman and Uzawa (1964) give the conditions characterized by Slutsky terms for the observed demand functions to be consistent with a strongly (weakly) separable utility function. Here we assume the contingent demand functions satisfy the conditions for weak separability, the question we ask is: when will a weakly separable utility function be of the particular form (2.23)? Proposition 2 below gives the necessary and sufficient conditions for the derived contingent demands to be compatible with this particular functional form.

Proposition 2. *Assume $A > 2$ and $S > 2$, and there exist complete contingent claims, which can be rationalized by a well-defined, weakly separable utility. Then this utility function is ordinally equivalent to a smooth ambiguity utility if and only if there exist strictly monotone functions $f : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$, $\gamma : \mathbb{R}_{++} \to \mathbb{R}_{++}$, and $G : \mathbb{R}^2 \to \mathbb{R}$ such that for $a \in \{1, ..., A\}$ and $s \in \{1, ..., S\}$, the observed contingent demands satisfy*

$$x_{as} = f(x_{a1}, k_{as}), \tag{2.27}$$

$$\sum_{s=1}^{S} \nu_{as} \int \frac{\gamma(x_{a1})}{f^{-1}_{x_{a1}}(x_{as})} dx_{as} = G\left(\sum_{s=1}^{S} \nu_{1s} \int \frac{\gamma(x_{11})}{f^{-1}_{x_{11}}(x_{1s})} dx_{1s}, K_a\right), \tag{2.28}$$

where $k_{as} =_{def} \frac{\nu_{as}p_{a1}}{\nu_{a1}p_{as}}$*, and* $K_a =_{def} \frac{\mu_a\nu_{a1}\gamma(x_{a1})p_{11}}{\mu_1\nu_{11}\gamma(x_{11})p_{a1}}$*. And functions* f *and* G *satisfy:* $x = f(x,1)$*,* $u(\boldsymbol{x}) = G\big(u(\boldsymbol{x}),1\big)$*, and* $\frac{\partial^2 G}{\partial x_{as}x_{a1}} = 0$.

Remark 8. Condition (2.27) holds for each risk state s, conditional on ambiguity state a. So it is a restriction on the relation between risk state consumption demand x_{as} and its relative risk neutral price k_{as}.

Remark 9. Condition (2.28) relates the expected utility in state a (expressed in terms of integration of demands) to its relative ambiguity neutral price K_a. The expected utility level depends on the absolute level of the utility index u, as indicated by γ function; however, the relative ambiguity neutral price is independent of the unit of u.

Lemma 1. *When* $S > 2$*, a twice continuously differentiable utility function is additively separable* $u(\boldsymbol{x}) = \sum_{s=1}^{S} u_s(x_s)$ *if and only if* $\frac{\partial u_s/u_1}{\partial u_i} = 0$ *for* $i, s \in \{2, 3, ..., S\}$*, and* $i \neq s$*.*

This lemma is used in Kubler, Selden, and Wei (2014), who contribute it to Samuelson (1947). However, it should be noted that we only focus on the case of one good per state, the *additively separability* is a special case of *strong separability* in Definition 1, and the proof can be found in Goldman and Uzawa (1964). Now we go to the proof of Proposition 2.

Proof. Necessity

Since the objective function (2.23) satisfies the regularity assumption, the following first order conditions are necessary and sufficient for charactering the optimal solution to problem (2.26):

F.O.C with respect to x_{as},

$$\mu_a \phi' \left(\sum_{s=1}^{S} \nu_{as} u(x_{as}) \right) \nu_{as} u'(x_{as}) = \lambda p_{as}, \tag{2.29}$$

F.O.C with respect to $x_{as'}$,

$$\mu_a \phi' \left(\sum_{s=1}^{S} \nu_{as} u(x_{as}) \right) \nu_{as'} u'(x_{as'}) = \lambda p_{as'}, \tag{2.30}$$

F.O.C with respect to $x_{a's}$,

$$\mu_{a'}\phi^{'}\left(\sum_{s=1}^{S}\nu_{a's}u(x_{a's})\right)\nu_{a's}u^{'}(x_{a's}) = \lambda p_{a's}, \tag{2.31}$$

where λ is the Lagrange multiplier.

From equations (2.29) and (2.30), we have

$$\frac{\nu_{as}u^{'}(x_{as})}{\nu_{as'}u^{'}(x_{as'})} = \frac{p_{as}}{p_{as'}}. \tag{2.32}$$

From equations (2.29) and (2.31), we have

$$\frac{\mu_a\phi^{'}\left(\sum_{s=1}^{S}\nu_{as}u(x_{as})\right)\nu_{as}u^{'}(x_{as})}{\mu_{a'}\phi^{'}\left(\sum_{s=1}^{S}\nu_{a's}u(x_{a's})\right)\nu_{a's}u^{'}(x_{a's})} = \frac{p_{as}}{p_{a's}}. \tag{2.33}$$

Equation (2.32) characterizes the marginal rate of substitution between consumptions within the same ambiguity state, and equation (2.33) characterizes the marginal rate of substitution between consumptions across ambiguity states.

Rearrange terms in equation (2.32), we have

$$u^{'}(x_{as}) = \frac{\nu_{a1}p_{as}}{\nu_{as}p_{a1}}u^{'}(x_{a1}). \tag{2.34}$$

Define $k_{as} = \frac{\nu_{as}p_{a1}}{\nu_{a1}p_{as}}$, and substitute into equation (2.34), we have

$$x_{as} = u^{'-1}\left(\frac{u^{'}(x_{a1})}{k_{as}}\right) = f(x_{a1}, k_{as}). \tag{2.35}$$

Monotonicity of function f follows from concavity of u (or monotonicity of $u^{'}$). Note that this functional form holds contingent on each ambiguity states; however, the function f itself is invariant across ambiguity states.

To derive equation (2.28), define $u^{'}(x) = \gamma(x)$, then concavity of u implies that $\gamma(x)$ is a decreasing function of x.

From equations (2.34) and (2.35), we have

$$u(x_{as}) = \int \frac{\gamma(x_{a1})}{f_{x_{a1}}^{-1}(x_{as})}dx_{as}. \tag{2.36}$$

From equation (2.33), we have

$$\phi^{'}\left(\sum_{s=1}^{S}\nu_{as}u(x_{as})\right)=\frac{\mu_1\nu_{1s}u^{'}(x_{1s})p_{as}}{\mu_a\nu_{as}u^{'}(x_{as})p_{1s}}\phi^{'}\left(\sum_{s=1}^{S}\nu_{1s}u(x_{1s})\right). \tag{2.37}$$

Use the relation in equation (2.34), and substitute $u^{'}(x_{1s})$ and $u^{'}(x_{as})$ into above equation (2.37), we have

$$\phi^{'}\left(\sum_{s=1}^{S}\nu_{as}u(x_{as})\right)=\frac{\mu_1\nu_{11}\gamma(x_{11})p_{a1}}{\mu_a\nu_{a1}\gamma(x_{a1})p_{11}}\phi^{'}\left(\sum_{s=1}^{S}\nu_{1s}u(x_{1s})\right). \tag{2.38}$$

Use equation (2.36), and define $K_a=\frac{\mu_a\nu_{a1}\gamma(x_{a1})p_{11}}{\mu_1\nu_{11}\gamma(x_{11})p_{a1}}$, we get

$$\begin{aligned}\sum_{s=1}^{S}\nu_{as}\int\frac{\gamma(x_{a1})}{f_{x_{a1}}^{-1}(x_{as})}dx_{as}&=\phi^{'-1}\left(\frac{\phi^{'}\big(\sum\limits_{s=1}^{S}\nu_{1s}\int\frac{\gamma(x_{11})}{f_{x_{11}}^{-1}(x_{1s})}dx_{1s}\big)}{K_a}\right)\\&=G\left(\sum_{s=1}^{S}\nu_{1s}\int\frac{\gamma(x_{11})}{f_{x_{11}}^{-1}(x_{1s})}dx_{1s},K_a\right).\end{aligned} \tag{2.39}$$

Denote $\sum\limits_{s=1}^{S}\nu_{as}\int\frac{\gamma(x_{a1})}{f_{x_{a1}}^{-1}(x_{as})}dx_{as}$ by $u(\boldsymbol{x}_a)$, then the follow holds

$$u(\boldsymbol{x})=G\big(u(\boldsymbol{x}),1\big). \tag{2.40}$$

Since $\phi^{'}$ is strictly monotone, the function G will be strictly monotone. If we take derivative w.r.t x_{as} on both sides of equation (2.39), we have

$$\frac{\partial G}{\partial x_{as}}=G_2\frac{\partial K_a}{\partial x_{as}}=\nu_{as}u^{'}(x_{as}), \tag{2.41}$$

which is a function of x_{as} only.

So we have

$$\frac{\partial^2 G}{\partial x_{as}\partial x_{a1}}=0 \tag{2.42}$$

Sufficiency

We assume that contingent claim demand functions satisfy income-Lipschitz condition, and the Slutsky matrix of contingent claim demands satisfies the conditions for weak separability as shown by Goldman and Uzawa (1964), so there exists a weakly separable function

$$\Phi\Big(..., u_a\big(x_{a1}, ..., x_{aS}; \boldsymbol{\mu}; \boldsymbol{\nu}\big), ...; \boldsymbol{\mu}; \boldsymbol{\nu}\Big), \tag{2.43}$$

which can rationalize observed contingent consumption.

So the observed contingent claim demands solves the following problem:

$$\max_{x\in\mathbb{R}^{AS}_{++}} \Phi\Big(..., u_a\big(x_{a1}, ..., x_{aS}; \boldsymbol{\mu}; \boldsymbol{\nu}\big), ...; \boldsymbol{\mu}; \boldsymbol{\nu}\Big) \ s.t. \ \boldsymbol{p}\cdot\boldsymbol{x} \leq I. \tag{2.44}$$

The question is whether it takes the smooth ambiguity utility form.

Step one: additive separability of u

First, we show that the condition $x_{as} = f(x_{a1}, k_{as})$ implies that u_a is additively separable across risk states conditional on certain ambiguity state.

From the first order condition for problem (2.44), we have

$$\frac{\partial u_a/\partial x_{a1}}{\partial u_a/\partial x_{as}} = \frac{p_{a1}}{p_{as}}. \tag{2.45}$$

Since $x_{as} = f(x_{a1}, k_{as})$, and $f(x_{a1}, k_{as})$ is a strictly monotone function of k_{as}, we have

$$\frac{\nu_{as}p_{a1}}{\nu_{a1}p_{as}} = k_{as} = f^{-1}_{x_{a1}}(x_{as}), \tag{2.46}$$

which implies that the first order condition (2.45) can be rewritten as

$$\frac{\partial u_a/\partial x_{a1}}{\partial u_a/\partial x_{as}} = \frac{\nu_{a1}}{\nu_{as}} f^{-1}_{x_{a1}}(x_{as}), \tag{2.47}$$

which implies that

$$\frac{\partial(\frac{\partial u_a/\partial x_{a1}}{\partial u_a/\partial x_{as}})}{\partial x_i} = 0 \ (i \neq s). \tag{2.48}$$

The result from Lemma 1 implies that we can assume u takes the form

$$u_a(\boldsymbol{x}_a;\boldsymbol{\mu};\boldsymbol{\nu}) = \sum_{s=s}^{S} u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu}). \tag{2.49}$$

The first order condition (2.45) now can be rewritten as

$$\frac{u'_{a1}(x_{a1};\boldsymbol{\mu};\boldsymbol{\nu})}{u'_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})} = \frac{p_{a1}}{p_{as}}. \tag{2.50}$$

Combine with equation (2.46), we have

$$\frac{\pi_{as}u'_{a1}(x_{a1};\boldsymbol{\mu};\boldsymbol{\nu})}{\pi_{a1}u'_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})} = k_{as} = f_{x_{a1}}^{-1}(x_{as}), \tag{2.51}$$

which must be independent of $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$.

Denoting

$$t_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu}) = \frac{u'_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}}. \tag{2.52}$$

We have

$$\frac{\partial}{\partial\boldsymbol{\nu}}\left(\frac{t_{a1}(x_{a1};\boldsymbol{\mu};\boldsymbol{\nu})}{t_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}\right) = 0, \tag{2.53}$$

$$\frac{\partial}{\partial\boldsymbol{\mu}}\left(\frac{t_{a1}(x_{a1};\boldsymbol{\mu};\boldsymbol{\nu})}{t_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}\right) = 0. \tag{2.54}$$

These imply that

$$\frac{\partial \ln t_{a1}(x_{a1};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial\boldsymbol{\nu}} = \frac{\partial \ln t_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial\boldsymbol{\nu}}, \tag{2.55}$$

$$\frac{\partial \ln t_{a1}(x_{a1};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial\boldsymbol{\mu}} = \frac{\partial \ln t_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial\boldsymbol{\mu}}. \tag{2.56}$$

Taking derivative with respect to x_{as} on both sides of the above equations yields

$$\frac{\partial^2 \ln t_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}\partial\boldsymbol{\nu}} = 0, \tag{2.57}$$

$$\frac{\partial^2 \ln t_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}\partial\boldsymbol{\mu}} = 0. \tag{2.58}$$

These imply that

$$\ln t_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu}) = t^1_{as}(x_{as}) + t^2_{as}(\boldsymbol{\mu};\boldsymbol{\nu}). \tag{2.59}$$

Define

$$\tau^1_{as}(x_{as}) = \exp\left(t^1_{as}(x_{as})\right), \tag{2.60}$$

$$\tau^2_{as}(\boldsymbol{\mu};\boldsymbol{\nu}) = \exp\left(t^2_{as}(\boldsymbol{\mu};\boldsymbol{\nu})\right). \tag{2.61}$$

We have

$$t_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu}) = \tau^1_{as}(x_{as})\tau^2_{as}(\boldsymbol{\mu};\boldsymbol{\nu}). \tag{2.62}$$

Due to equations (2.53) and (2.54), $\tau^2_{as}(\boldsymbol{\mu};\boldsymbol{\nu})$ must be the same across risk states, and can be denoted $\tau^2_a(\boldsymbol{\mu};\boldsymbol{\nu})$.

We have

$$u'_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu}) = \tau^2_a(\boldsymbol{\mu};\boldsymbol{\nu})\nu_{as}\tau^1_{as}(x_{as}). \tag{2.63}$$

Since when $k_{as} = 1$, $x_{as} = f(x_{a1}, 1) = x_{a1}$, we must have $\tau^1_{a1} = \tau^1_{as} = \tau^1_a$. Define

$$u_a(x_{as}) = \int \tau^1_a(x_{as})dx_{as}. \tag{2.64}$$

Finally since

$$f(x_{a1}, k_{as}) = u'^{-1}_a\left(\frac{u'_a(x_{a1})}{k_{as}}\right) \tag{2.65}$$

is strictly increasing in k_{as}, u_a is strictly concave.

Step two: additive separability of ϕ

First, we want to show that the condition (2.28) implies that Φ is additively separable across ambiguity states.

From the previous construction, we have

$$u_a(x_{a1},...x_{aS};\boldsymbol{\mu};\boldsymbol{\nu}) = \tau^2_a(\boldsymbol{\mu};\boldsymbol{\nu})\sum_{s=1}^{S}\nu_{as}u_a(x_{as}). \tag{2.66}$$

The objective function can be rewritten as

$$\Phi\left(..., \sum_{s=1}^{S} \nu_{as} u_a(x_{as}), ...; \boldsymbol{\mu}; \boldsymbol{\nu}\right). \tag{2.67}$$

The first order condition across ambiguity states would be

$$\frac{\frac{\partial \Phi}{\partial U_a} \nu_{as} u_a^{'}(x_{as})}{\frac{\partial \Phi}{\partial U_{a^{'}}} \nu_{a^{'}s} u_{a^{'}}^{'}(x_{a^{'}s})} = \frac{p_{as}}{p_{a^{'}s}}. \tag{2.68}$$

Since the contingent consumption is consistent with a utility function which is weakly separable across ambiguity states, from the proof in step one,

$$\frac{\nu_{as} u_a^{'}(x_{as})}{\nu_{a1} u_a^{'}(x_{a1})} = \frac{p_{as}}{p_{a1}}, \tag{2.69}$$

which implies $\sum_{s=1}^{S} \nu_{as} \int \frac{\gamma(x_{a1})}{f_{x_{a1}}^{-1}(x_{as})} dx_{as}$ is function of x_{a1},...,x_{aS} only, independent of contingent demands in other ambiguity states.

Since function G is strictly monotone, by condition (2.28), we have

$$K_a = G_{U(\boldsymbol{x}_1)}^{-1}\big(U(\boldsymbol{x}_a)\big). \tag{2.70}$$

The first order condition (2.68) can be rewritten as

$$\frac{\frac{\partial \Phi}{\partial U_a} \nu_{a1} u_a^{'}(x_{a1})}{\frac{\partial \Phi}{\partial U_1} \nu_{11} u_1^{'}(x_{11})} = \frac{\mu_1 \nu_{11} \gamma(x_{11})}{\mu_a \nu_{a1} \gamma(x_{a1})} K_a. \tag{2.71}$$

It implies that

$$\frac{\partial \frac{\frac{\partial \Phi}{\partial U_a} \nu_{a1} u_a^{'}(x_{a1})}{\frac{\partial \Phi}{\partial U_1} \nu_{11} u_1^{'}(x_{11})}}{\partial x_{it}} = 0, \text{ for } i \neq 1, a. \tag{2.72}$$

Thus we can assume Φ takes the form

$$\Phi\left(..., \sum_{s=1}^{S} \nu_{as} u_a(x_{as}), ...; \boldsymbol{\mu}; \boldsymbol{\nu}\right) = \sum_{a=1}^{A} \phi_a \left(\sum_{s=1}^{S} \nu_{as} u_a(x_{as}); \boldsymbol{\mu}; \boldsymbol{\nu}\right). \tag{2.73}$$

So the first order condition (2.68) will be

$$\frac{\phi_1'\left(\sum \nu_{1s}u_1(x_{1s});\boldsymbol{\mu};\boldsymbol{\nu}\right)\nu_{1s}u_1'(x_{1s})}{\phi_a'\left(\sum \nu_{as}u_a(x_{as});\boldsymbol{\mu};\boldsymbol{\nu}\right)\nu_{as}u_2'(x_{as})} = \frac{p_{1s}}{p_{as}}. \tag{2.74}$$

Equivalently,

$$\frac{\mu_a\gamma_a\phi_1'\left(\sum \nu_{1s}u_1(x_{1s});\boldsymbol{\mu};\boldsymbol{\nu}\right)\nu_{1s}\nu_{as}u_1'(x_{1s})}{\mu_1\gamma_1\phi_a'\left(\sum \nu_{as}u_a(x_{as});\boldsymbol{\mu};\boldsymbol{\nu}\right)\nu_{1s}\nu_{as}u_a'(x_{as})} = K_a = G_{U_1}^{-1}(U_a), \tag{2.75}$$

which must be independent of $\boldsymbol{\pi}_{a'}$ ($a' \neq a$ or 1) and $\boldsymbol{\mu}$.

The above equation (2.75) can be reduced to

$$\frac{\mu_a\gamma_a\phi_1'\left(\sum \nu_{1s}u_1(x_{1s});\boldsymbol{\mu};\boldsymbol{\nu}\right)u_1'(x_{1s})}{\mu_1\gamma_1\phi_a'\left(\sum \nu_{as}u_a(x_{as});\boldsymbol{\mu};\boldsymbol{\nu}\right)u_a'(x_{as})} = K_a = G_{U_1}^{-1}(U_a). \tag{2.76}$$

Denote

$$H_a\left(\sum \nu_{as}u_a(x_{as});\boldsymbol{\mu};\boldsymbol{\nu}\right) = \frac{\phi_a'\left(\sum \nu_{as}u_a(x_{as});\boldsymbol{\mu};\boldsymbol{\nu}\right)u'(x_{as})}{\mu_a\gamma_a}. \tag{2.77}$$

We have

$$\frac{\partial\left(\frac{H_1(\sum \nu_{1s}u_1(x_{1s});\boldsymbol{\mu};\boldsymbol{\nu})}{H_a(\sum \nu_{as}u_a(x_{as});\boldsymbol{\mu};\boldsymbol{\nu})}\right)}{\partial \boldsymbol{\nu}_{a'}} = 0, \tag{2.78}$$

$$\frac{\partial\left(\frac{H_1(\sum \nu_{1s}u_1(x_{1s});\boldsymbol{\mu};\boldsymbol{\nu})}{H_a(\sum \nu_{as}u_a(x_{as});\boldsymbol{\mu};\boldsymbol{\nu})}\right)}{\partial \boldsymbol{\mu}} = 0. \tag{2.79}$$

Equivalently,

$$\frac{\partial \ln H_1\left(\sum \nu_{1s}u_1(x_{1s});\boldsymbol{\mu};\boldsymbol{\nu}\right)}{\partial \boldsymbol{\nu}_{a'}} = \frac{\partial \ln H_a\left(\sum \nu_{as}u_a(x_{as});\boldsymbol{\mu};\boldsymbol{\nu}\right)}{\partial \boldsymbol{\nu}_{a'}}, \tag{2.80}$$

$$\frac{\partial \ln H_1\bigg(\sum \nu_{1s}u_1(x_{1s});\boldsymbol{\mu};\boldsymbol{\nu}\bigg)}{\partial \boldsymbol{\mu}}=\frac{\partial \ln H_a\bigg(\sum \nu_{as}u_a(x_{as});\boldsymbol{\mu};\boldsymbol{\nu}\bigg)}{\partial \boldsymbol{\mu}}. \tag{2.81}$$

Take derivative with respect to x_{as}, we have

$$\frac{\partial^2 \ln H_a\bigg(\sum \nu_{as}u_a(x_{as});\boldsymbol{\mu};\boldsymbol{\nu}\bigg)}{\partial x_{as}\partial \boldsymbol{\nu}_{a'}}=0, \tag{2.82}$$

$$\frac{\partial^2 \ln H_a\bigg(\sum \nu_{as}u_a(x_{as});\boldsymbol{\mu};\boldsymbol{\nu}\bigg)}{\partial x_{as}\partial \boldsymbol{\mu}}=0, \tag{2.83}$$

which implies that

$$\ln H_a\bigg(\sum \nu_{as}u_g(x_{as});\boldsymbol{\mu};\boldsymbol{\nu}\bigg)=\tau_a^1\bigg(\sum \nu_{as}u_a(x_{as})\bigg)+\tau_a^2\bigg(\boldsymbol{\mu},\boldsymbol{\nu}_{-a}\bigg). \tag{2.84}$$

Let

$$\Gamma_a^1\bigg(\sum \nu_{as}u_a(x_{as})\bigg)=\exp\bigg(\tau_a^1\big(\sum \nu_{as}u_a(x_{as})\big)\bigg), \tag{2.85}$$

$$\Gamma_a^2\bigg(\boldsymbol{\nu}_{-a},\boldsymbol{\mu}\bigg)=\exp\bigg(\tau_a^2\big(\boldsymbol{\mu},\boldsymbol{\nu}_{-a}\big)\bigg). \tag{2.86}$$

Then

$$H_a\bigg(\sum \nu_{as}u_a(x_{as});\boldsymbol{\mu};\boldsymbol{\nu}\bigg)=\Gamma_a^1\bigg(\sum \nu_{as}u_a(x_{as})\bigg)\Gamma_a^2\bigg(\boldsymbol{\mu},\boldsymbol{\nu}_{-a}\bigg). \tag{2.87}$$

Due to equations (2.78) and (2.79), $\Gamma_a^2\left(\boldsymbol{\mu},\boldsymbol{\nu}_{-a}\right)$ must be independent of $\boldsymbol{\nu}_{-a}$, and be the same for all $a=1,2...,A$. So it can be denoted by $\Gamma^2(\boldsymbol{\mu})$.

And from the proof in step one, we have

$$\frac{u_a^{'}(x_{as})}{u_a^{'}(x_{a1})}=\frac{\nu_{a1}p_{as}}{\nu_{as}p_{a1}}=\frac{1}{f_{x_{a1}}^{-1}(x_{as})}. \tag{2.88}$$

So the equation (2.28) can be rewritten as

$$\sum_{s=1}^{S} \nu_{as} \int \frac{u_a^{'}(x_{as})}{u_a^{'}(x_{a1})} \gamma(x_{a1}) dx_{as} = G\left(\sum_{s=1}^{S} \nu_{1s} \int \frac{u_1^{'}(x_{1s})}{u_1^{'}(x_{11})} \gamma(x_{11}) dx_{1s}, K_a\right). \tag{2.89}$$

Take derivative w.r.t x_{as} on both sides of the equation (2.89), we have

$$\nu_{as} \frac{u_a^{'}(x_{as})}{u_a^{'}(x_{a1})} \gamma(x_{a1}) = \frac{\partial G}{\partial x_{as}}. \tag{2.90}$$

Since $\frac{\partial^2 G}{\partial x_{as} \partial x_{a1}} = 0$, the left-hand side of the above equation should be independent of x_{a1}. Therefore, we have

$$\gamma(x_{a1}) = u_a^{'}(x_{a1}). \tag{2.91}$$

When $K_a = 1$,

$$\sum \nu_{1s} u(x_{1s}) = \sum \nu_{as} u(x_{as}). \tag{2.92}$$

Equation (2.68) implies

$$\frac{\mu_a \gamma(x_{a1}) \Gamma^2(\boldsymbol{\mu}) \mu_1 \gamma(x_{11}) \Gamma_1^1\left(\sum \nu_{1s} u(x_{1s})\right)}{\mu_1 \gamma(x_{11}) \Gamma^2(\boldsymbol{\mu}) \mu_a \gamma(x_{a1}) \Gamma_a^1\left(\sum \nu_{as} u(x_{as})\right)} = 1. \tag{2.93}$$

Therefore

$$\Gamma_1^1\left(\sum \nu_{1s} u(x_{1s})\right) = \Gamma_a^1\left(\sum \nu_{as} u(x_{as})\right). \tag{2.94}$$

So

$$\Gamma_1^1 = \Gamma_a^1 = \Gamma^1. \tag{2.95}$$

Define

$$\phi(u) = \int \Gamma^1(u) du. \tag{2.96}$$

Since the first order condition implies

$$\frac{\mu_a \gamma(x_{a1})\phi_1^{'}\left(\sum \nu_{1s}u(x_{1s})\right)u^{'}(x_{1s})}{\mu_1 \gamma(x_{11})\phi_a^{'}\left(\sum \nu_{as}u(x_{as})\right)u^{'}(x_{as})} = \frac{\mu_a p_{1s}\nu_{as}\gamma(x_{a1})}{\mu_1 p_{as}\nu_{1s}\gamma(x_{11})}. \tag{2.97}$$

Use the condition $\lambda(x_{a1}) = u^{'}(x_{a1})$ and let $s = 1$, the above equation will be reduced to

$$\frac{\mu_a \phi_1^{'}\left(\sum \nu_{1s}u(x_{1s})\right)}{\mu_1 \phi_a^{'}\left(\sum \nu_{as}u(x_{as})\right)} = \frac{\mu_a p_{1s}\nu_{as}\gamma(x_{a1})}{\mu_1 p_{as}\nu_{1s}\gamma(x_{11})}. \tag{2.98}$$

Substitute $\phi_a^{'}\left(\sum \nu_{as}u(x_{as})\right) = \Gamma^2(\boldsymbol{\mu})\mu_a \Gamma^1\left(\sum \nu_{as}u(x_{as})\right)$, we have

$$\frac{\Gamma^1\left(\sum \nu_{1s}u(x_{1s})\right)}{\Gamma^1\left(\sum \nu_{as}u(x_{as})\right)} = \frac{\mu_a p_{1s}\nu_{as}\gamma(x_{a1})}{\mu_1 p_{as}\nu_{1s}\gamma(x_{11})}. \tag{2.99}$$

So

$$\sum \nu_{as}u(x_{as}) = \phi^{'-1}\left(\frac{\phi^{'}(\sum \nu_{1s}u(x_{1s}))}{\frac{\mu_a p_{1s}\nu_{as}\gamma(x_{a1})}{\mu_1 p_{as}\nu_{1s}\gamma(x_{11})}}\right). \tag{2.100}$$

which implies that ϕ is strictly concave. $\square$

2.3.2 Demand derivative test

In the functional form test, we assume the Slutsky matrix derived from contingent consumption demands is consistent with a weakly separable utility, and then give the necessary and sufficient conditions for these contingent consumption demands to be rationalized by some smooth ambiguity utility function. These necessary and sufficient conditions are expressed by the relation between contingent demand and relative risk neutral price, and the relation between contingent expected utility and relative ambiguity neutral price.

In this section, we derive the necessary and sufficient conditions on Slutsky terms. Notice that the objective function is not just weakly separable, but also strongly separable. The Slutsky conditions we derive will characterize strong separability, stationarity of utility indices, and homogeneity in probability.

Given the regularity assumption, the optimal solution to problem (2.26) is determined by $\boldsymbol{F}(\boldsymbol{x}, \lambda, \boldsymbol{p}, I, \boldsymbol{\mu}, \boldsymbol{\nu}) = \mathbf{0}$, where

$$\boldsymbol{F}(\boldsymbol{x}, \lambda, \boldsymbol{p}, I, \boldsymbol{\mu}, \boldsymbol{\nu}) = \begin{cases} D_{\boldsymbol{x}} U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu}) - \lambda \boldsymbol{p} = \mathbf{0} \\ I - \boldsymbol{p}\boldsymbol{x} = 0 \end{cases} . \tag{2.101}$$

Under Assumption $1''$, the Hessian matrix $\boldsymbol{D}^2_{\boldsymbol{xx}} U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})$ is negative definite, so that $\boldsymbol{D}_{\boldsymbol{x},\lambda} \boldsymbol{F}(\boldsymbol{x}, \lambda, \boldsymbol{p}, I, \boldsymbol{\mu}, \boldsymbol{\nu}) = \mathbf{0}$ is invertible.

Define the inverse matrix as

$$\begin{bmatrix} \boldsymbol{K} & -\boldsymbol{\zeta} \\ -\boldsymbol{\zeta}^T & b \end{bmatrix} = \begin{bmatrix} \boldsymbol{D}^2_{\boldsymbol{xx}} U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu}) & -\boldsymbol{p} \\ -\boldsymbol{p}^T & 0 \end{bmatrix}^{-1}, \tag{2.102}$$

where $\boldsymbol{K}$ is a symmetric $AS \times AS$ matrix and $\boldsymbol{\zeta}$ is a $AS \times 1$ vector.

Defining

$$\boldsymbol{\Sigma} = \lambda \boldsymbol{K}. \tag{2.103}$$

Apply the implicit function theorem, we have

$$\boldsymbol{D}_{\boldsymbol{p},I,\boldsymbol{\mu},\boldsymbol{\nu}}(\boldsymbol{x}, \lambda) = -(\boldsymbol{D}_{\boldsymbol{x},\lambda}\boldsymbol{F})^{-1} \boldsymbol{D}_{\boldsymbol{p},I,\boldsymbol{\mu},\boldsymbol{\nu}} \boldsymbol{F}. \tag{2.104}$$

Therefore we have

$$\boldsymbol{D}_{\boldsymbol{p}} \boldsymbol{x} = \boldsymbol{\Sigma} - \boldsymbol{\zeta} \boldsymbol{x}^T, \tag{2.105}$$

$$\boldsymbol{D}_I \boldsymbol{x} = \boldsymbol{\zeta}, \tag{2.106}$$

$$\boldsymbol{D}_{\boldsymbol{\mu}} \boldsymbol{x} = -\boldsymbol{K} \boldsymbol{D}^2_{\boldsymbol{\mu x}} U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu}), \tag{2.107}$$

$$\boldsymbol{D}_{\boldsymbol{\nu}} \boldsymbol{x} = -\boldsymbol{K} \boldsymbol{D}^2_{\boldsymbol{\nu x}} U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu}). \tag{2.108}$$

The symmetric matrix $\boldsymbol{\Sigma} = \left(\sigma_{as,a's'}\right)_{AS \times AS}$ has rank $AS - 1$, and satisfies $\boldsymbol{\Sigma} \boldsymbol{p} = \mathbf{0}$.

The following proposition gives necessary and sufficient conditions for the contingent demands to be rationalized by some smooth ambiguity preference. These conditions involve restrictions on the terms in matrices $\boldsymbol{D_p x}$, $\boldsymbol{D_I x}$, $\boldsymbol{D_\mu x}$, and $\boldsymbol{D_\nu x}$.

Proposition 3. *When $A > 2$ and $S > 2$, contingent consumption demands can be rationalized by a state independent smooth ambiguity utility function if and only if the following conditions hold:*

$$\frac{\sigma_{as,a't}}{\zeta_{as}\zeta_{a't}} = \frac{\sigma_{as',a't'}}{\zeta_{as'}\zeta_{a't'}} = H(\boldsymbol{x}), \tag{2.109}$$

for some function $H(\boldsymbol{x})$, $a \neq a'$;

$$\frac{\sigma_{as,as'}}{\zeta_{as}\zeta_{as'}} = \frac{\sigma_{at,at'}}{\zeta_{at}\zeta_{at'}} = H^a(\boldsymbol{x}), \tag{2.110}$$

for some function $H^a(\boldsymbol{x})$;

$$\frac{\partial x_{as}}{\partial \mu_{a'}} = \frac{\sum_{s=1}^{S} \sigma_{as,a's} p_{a's}}{\mu_{a'}}; \tag{2.111}$$

$$\frac{\partial x_{at}}{\partial \nu_{as}} - \frac{\partial x_{at}}{\partial \nu_{as'}} = (\sigma_{at;as} - \sigma_{at;as'})\frac{p_{as}}{\pi_{as}} \text{ at } x_{as} = x_{as'}; \tag{2.112}$$

$$x_{as} = x_{as'} \text{ only if } \frac{p_{as}}{\nu_{as}} = \frac{p_{as'}}{\nu_{as'}}; \tag{2.113}$$

$$x_{as} = x_{as'} = x_{a's} = x_{a's'} \text{ only if } \frac{p_{as}}{\mu_a \nu_{as}} = \frac{p_{a's}}{\mu_{a'}\nu_{a's}}. \tag{2.114}$$

Proof. Necessity

Condition (2.109) for strong separability across ambiguity states, and condition (2.110) for strong separability across risk states conditional on each ambiguity state follow from Goldman and Uzawa (1964).

Condition (2.113) follows from stationarity of index u, and condition (2.114) follows from stationarity of index ϕ.

Now, let's focus on terms in matrix $\boldsymbol{D}^2_{\boldsymbol{\mu x}} U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})$, which has dimension of $AS \times A$. In the first column of this matrix, the first S terms are non-zero with typical element $\lambda \frac{p_{1s}}{\mu_1}$, and the remaining terms are zeros. In the *ath* column, the terms from $a1$ to aS are non-zeros with typical element $\lambda \frac{p_{as}}{\mu_a}$, and the remaining terms are zeros.

So the matrix $-\boldsymbol{K D}^2_{\boldsymbol{\mu x}} U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})$ would have dimension of $AS \times A$

with first-row first-column element $\frac{\sum_{s=1}^{S}\sigma_{11,1s}p_{1s}}{\mu_1}$, first-row second-column element $\frac{\sum_{s=1}^{S}\sigma_{11,2s}p_{2s}}{\mu_2}$, first-row last-column element $\frac{\sum_{s=1}^{S}\sigma_{11,As}p_{As}}{\mu_A}$; second-row first-column element $\frac{\sum_{s=1}^{S}\sigma_{12,1s}p_{1s}}{\mu_1}$. The typical element of this matrix is $\frac{\sum_{s=1}^{S}\sigma_{as,a's}p_{a's}}{\mu_{a'}}$. Therefore we have the following restriction:

$$\frac{\partial x_{as}}{\partial \mu_{a'}} = \frac{\sum_{s=1}^{S}\sigma_{as,a's}p_{a's}}{\mu_{a'}}. \tag{2.115}$$

Next, let's look at the matrix $\boldsymbol{D}^2_{\boldsymbol{\nu x}}U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})$, which has dimension of $AS \times AS$. Since $\frac{\partial U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{11}} = \mu_1\phi'\left(\sum_{s=1}^{S}\nu_{1s}u(x_{1s})\right)\nu_{11}u'(x_{11})$, in the first row of $\boldsymbol{D}_{\boldsymbol{\nu x}}U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})$, the first S elements are nonzero, and the other elements are zero since $\mu_1\phi'\left(\sum_{s=1}^{S}\nu_{1s}u(x_{1s})\right)\nu_{11}u'(x_{11})$ is not function of ν_{as} for $a \neq 1$. For the non-zero elements: the first row-first column element is $\mu_1\phi''\left(\sum_{s=1}^{S}\nu_{1s}u(x_{1s})\right)\nu_{11}u'(x_{11})u(x_{11}) + \mu_1\phi'\left(\sum_{s=1}^{S}\nu_{1s}u(x_{1s})\right)u'(x_{11})$, and the first row-$sth$ column element is $\mu_1\phi''\left(\sum_{s=1}^{S}\nu_{1s}u(x_{1s})\right)\nu_{11}u'(x_{11})u'(x_{1s})$, where $s \neq 1$.

So the matrices $\boldsymbol{D}_{\boldsymbol{\nu}}\boldsymbol{x}$, $\boldsymbol{K}$ and $\boldsymbol{D}^2_{\boldsymbol{\nu x}}U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})$ have the following structure with corresponding typical elements;

$$\boldsymbol{D}_{\boldsymbol{\nu}}\boldsymbol{x} = \begin{bmatrix} \frac{\partial x_{11}}{\partial \nu_{11}} & \frac{\partial x_{11}}{\partial \nu_{12}} & \cdots & \frac{\partial x_{11}}{\partial \nu_{1S}} & \cdots & \frac{\partial x_{11}}{\partial \nu_{a1}} & \frac{\partial x_{11}}{\partial \nu_{a2}} & \cdots & \frac{\partial x_{11}}{\partial \nu_{aS}} & \cdots \\ \frac{\partial x_{12}}{\partial \nu_{11}} & \frac{\partial x_{12}}{\partial \nu_{12}} & \cdots & \frac{\partial x_{12}}{\partial \nu_{1S}} & \cdots & \frac{\partial x_{12}}{\partial \nu_{a1}} & \frac{\partial x_{12}}{\partial \nu_{a2}} & \cdots & \frac{\partial x_{12}}{\partial \nu_{aS}} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_{AS}}{\partial \nu_{11}} & \frac{\partial x_{AS}}{\partial \nu_{12}} & \cdots & \frac{\partial x_{AS}}{\partial \nu_{1S}} & \cdots & \frac{\partial x_{AS}}{\partial \nu_{a1}} & \frac{\partial x_{AS}}{\partial \nu_{a2}} & \cdots & \frac{\partial x_{AS}}{\partial \nu_{aS}} & \cdots \end{bmatrix}, \tag{2.116}$$

$$\boldsymbol{K} = \begin{bmatrix} \sigma_{11;11} & \cdots & \sigma_{11;1S} & \cdots & \sigma_{11;a1} & \cdots & \sigma_{11;aS} & \cdots \\ \sigma_{12;11} & \cdots & \sigma_{12;1S} & \cdots & \sigma_{12;a1} & \cdots & \sigma_{12;aS} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{AS;11} & \cdots & \sigma_{AS;1S} & \cdots & \sigma_{AS;a1} & \cdots & \sigma_{AS;aS} & \cdots \end{bmatrix}, \tag{2.117}$$

$$
\boldsymbol{D}^2_{\boldsymbol{\nu x}} U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu}) = \begin{bmatrix} \frac{\partial^2 U}{\partial x_{11}\partial \nu_{11}} & \dots & \frac{\partial^2 U}{\partial x_{11}\partial \nu_{1S}} & \dots & \frac{\partial^2 U}{\partial x_{11}\partial \nu_{a1}} & \dots \\ \frac{\partial^2 U}{\partial x_{12}\partial \nu_{11}} & \dots & \frac{\partial^2 U}{\partial x_{12}\partial \nu_{1S}} & \dots & \frac{\partial^2 U}{\partial x_{12}\partial \nu_{a1}} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial^2 U}{\partial x_{AS}\partial \nu_{11}} & \dots & \frac{\partial^2 U}{\partial x_{AS}\partial \nu_{1S}} & \dots & \frac{\partial^2 U}{\partial x_{AS}\partial \nu_{a1}} & \dots \end{bmatrix}
$$

$$
= \begin{bmatrix} \mu_1 \phi^{''} \nu_{11} u^{'}(x_{11}) u(x_{11}) + \mu_1 \phi^{'} u^{'}(x_{11}) & \dots & \dots \\ \mu_1 \phi^{''} \nu_{12} u^{'}(x_{12}) u(x_{11}) & \dots & \dots \\ \dots & \dots & \dots \\ 0 & \dots & \dots \end{bmatrix}. \tag{2.118}
$$

We define a new matrix which has dimension of $AS \times A(S-1)$ as follows:

$$
\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ 0 & 0 & 0 & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots \end{bmatrix}. \tag{2.119}
$$

The defined matrix $\boldsymbol{M}$ has full column rank. By matrix multiplication, we have:

$$
\boldsymbol{D}^2_{\boldsymbol{\nu x}} U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu}) \boldsymbol{M} = \begin{bmatrix} \mu_1 \phi^{'} u^{'}(x_{11}) & 0 & \dots & 0 & \dots \\ -\mu_1 \phi^{'} u^{'}(x_{12}) & \mu_1 \phi^{'} u^{'}(x_{12}) & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots \end{bmatrix}
$$

$$
= \begin{bmatrix} \lambda \frac{p_{11}}{\nu_{11}} & 0 & \dots & 0 & \dots \\ -\lambda \frac{p_{12}}{\nu_{12}} & \lambda \frac{p_{12}}{\nu_{12}} & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots \end{bmatrix}, \tag{2.120}
$$

$$\boldsymbol{K}\boldsymbol{D}^2_{\boldsymbol{\nu}\boldsymbol{x}}U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})\boldsymbol{M} = \boldsymbol{K}\begin{bmatrix} \lambda\frac{p_{11}}{\nu_{11}} & 0 & \dots & 0 & \dots \\ -\lambda\frac{p_{12}}{\nu_{12}} & \lambda\frac{p_{12}}{\nu_{12}} & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots \end{bmatrix}$$
$$= \begin{bmatrix} \sigma_{11;11}\frac{p_{11}}{\nu_{11}} - \sigma_{11;12}\frac{p_{12}}{\nu_{12}} & \sigma_{11;12}\frac{p_{12}}{\nu_{12}} - \sigma_{11;13}\frac{p_{13}}{\nu_{13}} & \dots \\ \sigma_{12;11}\frac{p_{11}}{\nu_{11}} - \sigma_{12;12}\frac{p_{12}}{\nu_{12}} & \sigma_{12;12}\frac{p_{12}}{\nu_{12}} - \sigma_{12;13}\frac{p_{13}}{\nu_{13}} & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}, \tag{2.121}$$

$$\boldsymbol{D}_{\boldsymbol{\nu}}\boldsymbol{x}\boldsymbol{M} = \begin{bmatrix} \frac{\partial x_{11}}{\partial \nu_{11}} - \frac{\partial x_{11}}{\partial \nu_{12}} & \frac{\partial x_{11}}{\partial \nu_{12}} - \frac{\partial x_{11}}{\partial \nu_{13}} & \dots & \frac{\partial x_{11}}{\partial \nu_{1S-1}} - \frac{\partial x_{11}}{\partial \nu_{1S}} & \dots \\ \frac{\partial x_{12}}{\partial \nu_{11}} - \frac{\partial x_{12}}{\partial \nu_{12}} & \frac{\partial x_{12}}{\partial \nu_{12}} - \frac{\partial x_{12}}{\partial \nu_{13}} & \dots & \frac{\partial x_{12}}{\partial \nu_{1S-1}} - \frac{\partial x_{12}}{\partial \nu_{1S}} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial x_{AS}}{\partial \nu_{11}} - \frac{\partial x_{AS}}{\partial \nu_{12}} & \frac{\partial x_{AS}}{\partial \nu_{12}} - \frac{\partial x_{AS}}{\partial \nu_{13}} & \dots & \frac{\partial x_{AS}}{\partial \nu_{AS-1}} - \frac{\partial x_{AS}}{\partial \nu_{1S}} & \dots \end{bmatrix}. \tag{2.122}$$

Therefore we have the following restriction:

$$\frac{\partial x_{at}}{\partial \nu_{as}} - \frac{\partial x_{at}}{\partial \nu_{as'}} = (\sigma_{at;as} - \sigma_{at;as'})\frac{p_{as}}{\nu_{as}} = (\sigma_{at;as} - \sigma_{at;as'})\frac{p_{as'}}{\nu_{as'}}. \tag{2.123}$$

Sufficiency

If conditions (2.109) and (2.110) are satisfied, from Goldman and Uzawa (1964), we can assume that the utility function has the following form:

$$U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu}) = \sum_{a=1}^{A}\phi_a\left(\sum_{s=1}^{S}u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu});\boldsymbol{\mu};\boldsymbol{\nu}\right), \tag{2.124}$$

which implies

$$\frac{\partial U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}} = \phi_a^{'}\left(\sum_{s=1}^{S}u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu});\boldsymbol{\mu};\boldsymbol{\nu}\right)u_{as}^{'}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu}). \tag{2.125}$$

Condition (2.112) uniquely determines the matrix $\boldsymbol{D}_{\boldsymbol{\nu}}\boldsymbol{x}(\boldsymbol{p}, I, \boldsymbol{\mu}, \boldsymbol{\nu})\boldsymbol{M}$

at $x_{as} = x_{as'}$. Since $\boldsymbol{\Sigma}$ has rank $AS-1$, and $\boldsymbol{\Sigma x} = \mathbf{0}$ only if $\boldsymbol{x} = c\boldsymbol{p}$ for some constant c. All solutions to equation $\boldsymbol{D_{\nu} x}(\boldsymbol{p}, I, \boldsymbol{\mu}, \boldsymbol{\nu})\boldsymbol{M} = -\boldsymbol{K D^2_{\nu x}} U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})\boldsymbol{M}$ must have the form

$$\boldsymbol{D^2_{\nu x}} U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})\boldsymbol{M} = \begin{bmatrix} c_{11}p_{11} + \lambda\frac{p_{11}}{\nu_{11}} & c_{12}p_{11} & \cdots \\ c_{11}p_{12} - \lambda\frac{p_{12}}{\nu_{12}} & c_{12}p_{12} + \lambda\frac{p_{12}}{\nu_{12}} & \cdots \\ c_{11}p_{13} & c_{12}p_{13} - \lambda\frac{p_{13}}{\nu_{13}} & \cdots \\ c_{11}p_{14} & c_{12}p_{14} & \cdots \\ \cdots & \cdots & \cdots \\ c_{11}p_{as} & c_{12}p_{as} & \cdots \\ \cdots & \cdots & \cdots \\ c_{11}p_{AS} & c_{12}p_{AS} & \cdots \end{bmatrix}. \tag{2.126}$$

So generally, we have the following solution

$$\frac{\partial^2 U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})}{\partial \nu_{as} \partial x_{as}} - \frac{\partial^2 U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})}{\partial \nu_{at} \partial x_{as}} = c_{as}p_{as} + \lambda\frac{p_{as}}{\nu_{as}}, \tag{2.127}$$

$$\frac{\partial^2 U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})}{\partial \nu_{a't} \partial x_{as}} - \frac{\partial^2 U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})}{\partial \nu_{a't'} \partial x_{as}} = c_{a't}p_{as} \text{ for } s \neq t, t'. \tag{2.128}$$

Given the objective function has the form,

$$U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu}) = \sum_{a=1}^{A} \phi_a\left(\sum_{s=1}^{S} u_{as}(x_{as}; \boldsymbol{\mu}; \boldsymbol{\nu}); \boldsymbol{\mu}; \boldsymbol{\nu}\right). \tag{2.129}$$

We have the following F.O.C

$$\frac{u'_{as}(x_{as}; \boldsymbol{\mu}; \boldsymbol{\nu})}{u'_{as'}(x_{as'}; \boldsymbol{\mu}; \boldsymbol{\nu})} = \frac{p_{as}}{p_{as'}}, \tag{2.130}$$

$$\frac{\phi'_a\left(\sum_{s=1}^{S} u_{as}(x_{as}; \boldsymbol{\mu}; \boldsymbol{\nu}); \boldsymbol{\mu}; \boldsymbol{\nu}\right) u'_{as}(x_{as}; \boldsymbol{\mu}; \boldsymbol{\nu})}{\phi'_{a'}\left(\sum_{s=1}^{S} u_{a's}(x_{a's}; \boldsymbol{\mu}; \boldsymbol{\nu}); \boldsymbol{\mu}; \boldsymbol{\nu}\right) u'_{a's}(x_{a's'}; \boldsymbol{\mu}; \boldsymbol{\nu})} = \frac{p_{as}}{p_{a's}}. \tag{2.131}$$

Because

$$\frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}}}{\partial \nu_{as} \partial x_{as}} = \frac{1}{\nu_{as}} \frac{\partial^2 U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \nu_{as} \partial x_{as}} - \frac{1}{\nu_{as}^2} \frac{\partial U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}}, \tag{2.132}$$

$$\frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}}}{\partial \nu_{as'} \partial x_{as}} = \frac{1}{\nu_{as}} \frac{\partial^2 U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \nu_{as'} \partial x_{as}}. \tag{2.133}$$

Then we have

$$\frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}}}{\partial \nu_{as'} \partial x_{as}} - \frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}}}{\partial \nu_{at} \partial x_{as}} = c_{as'} \frac{p_{as}}{\nu_{as}}. \tag{2.134}$$

Therefore

$$\frac{\frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}}}{\partial \nu_{as'} \partial x_{as}} - \frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}}}{\partial \nu_{at} \partial x_{as}}}{\frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{ar}}}{\partial \nu_{as'} \partial x_{ar}} - \frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{ar}}}{\partial \nu_{at} \partial x_{ar}}} = \frac{\nu_{ar} p_{as}}{\nu_{as} p_{ar}}. \tag{2.135}$$

It follows from equations (2.130) and (2.131) that

$$\frac{p_{as}}{p_{ar}} = \frac{\frac{\partial U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}}}{\frac{\partial U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{ar}}} \Longleftrightarrow \frac{\nu_{ar} p_{as}}{\nu_{as} p_{ar}} = \frac{\frac{\partial (U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})/\nu_{as})}{\partial x_{as}}}{\frac{\partial (U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})/\nu_{ar})}{\partial x_{ar}}}. \tag{2.136}$$

Hence we have

$$\frac{\frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}}}{\partial \nu_{as'} \partial x_{as}} - \frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}}}{\partial \nu_{at} \partial x_{as}}}{\frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{ar}}}{\partial \nu_{as'} \partial x_{ar}} - \frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{ar}}}{\partial \nu_{at} \partial x_{ar}}} = \frac{\frac{\partial (U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})/\nu_{as})}{\partial x_{as}}}{\frac{\partial (U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})/\nu_{ar})}{\partial x_{ar}}}. \tag{2.137}$$

Since we have,

$$\frac{\partial U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}} = \phi_a^{'}\left(\sum_{s=1}^{S} u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu});\boldsymbol{\mu};\boldsymbol{\nu}\right)\frac{\partial u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}}, \tag{2.138}$$

$$\begin{aligned}\frac{\partial^2 U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \nu_{as'}\partial x_{as}} &= \frac{\partial \phi_a^{'}\left(\sum_{s=1}^{S} u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu});\boldsymbol{\mu};\boldsymbol{\nu}\right)}{\partial \nu_{as'}}\frac{\partial u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}}\\ &+ \phi_a^{''}\left(\sum_{s=1}^{S} u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu});\boldsymbol{\mu};\boldsymbol{\nu}\right)\frac{\partial u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}}\\ &\times \sum_{s=1}^{S}\frac{\partial u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \nu_{as'}}\\ &+ \phi_a^{'}\left(\sum_{s=1}^{S} u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu});\boldsymbol{\mu};\boldsymbol{\nu}\right)\frac{\partial^2 u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \nu_{as'}\partial x_{as}}.\end{aligned} \tag{2.139}$$

The nominator in the left hand side of equation (2.137) is

$$\begin{aligned}&\left(\frac{\partial \phi_a^{'}(\sum_{s=1}^{S} u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu});\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \nu_{as'}} - \frac{\partial \phi_a^{'}(\sum_{s=1}^{S} u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu});\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \nu_{at}}\right)\\ &\times \frac{\partial u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}}\\ &+\left(\sum_{s=1}^{S}\frac{\partial u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \nu_{as'}} - \sum_{s=1}^{S}\frac{\partial u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \nu_{at}}\right)\phi_a^{''}(\sum_{s=1}^{S} u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu}))\\ &\times \frac{\partial u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}}\\ &+\left(\frac{\partial^2 u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \nu_{as'}\partial x_{as}} - \frac{\partial^2 u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \nu_{at}\partial x_{as}}\right)\phi_a^{'}(\sum_{s=1}^{S} u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})).\end{aligned} \tag{2.140}$$

We would have

$$\frac{\frac{\partial^2(\frac{u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}})}{\partial \nu_{as'}\partial x_{as}}}{\frac{\partial(\frac{u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}})}{\partial x_{as}}} - \frac{\frac{\partial^2(\frac{u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}})}{\partial \nu_{at}\partial x_{as}}}{\frac{\partial(\frac{u_{as}(x_{as};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{as}})}{\partial x_{as}}} = \frac{\frac{\partial^2(\frac{u_{ar}(x_{ar};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{ar}})}{\partial \nu_{as'}\partial x_{ar}}}{\frac{\partial(\frac{u_{ar}(x_{ar};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{ar}})}{\partial x_{ar}}} - \frac{\frac{\partial^2(\frac{u_{ar}(x_{ar};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{ar}})}{\partial \nu_{at}\partial x_{ar}}}{\frac{\partial(\frac{u_{ar}(x_{ar};\boldsymbol{\mu};\boldsymbol{\nu})}{\nu_{ar}})}{\partial x_{ar}}}. \tag{2.141}$$

Define

$$H_{as} = \frac{\partial(\frac{u_{as}}{\nu_{as}})}{\partial x_{as}}. \tag{2.142}$$

Then equation (2.141) implies that

$$\frac{\partial \ln H_{as}(x_{as})}{\partial \nu_{as'}} - \frac{\partial \ln H_{as}(x_{as})}{\partial \nu_{at}} = \frac{\partial \ln H_{ar}(x_{ar})}{\partial \nu_{as'}} - \frac{\partial \ln H_{ar}(x_{ar})}{\partial \nu_{at}}. \tag{2.143}$$

Take derivatives with respect to x_{as}, we have

$$\frac{\partial^2 \ln H_{as}(x_{as})}{\partial \nu_{as'} \partial x_{as}} = \frac{\partial \ln H_{as}(x_{as})}{\partial \nu_{at} \partial x_{as}}. \tag{2.144}$$

We can also derive such equation with respect to probability in ambiguity state $a^{'}$ (i.e. $\nu_{a^{'}s}$).

The general solution to this partial differential equation (2.144) should be

$$\begin{aligned}\ln H_{as}(x_{as}) &= f(x_{as}, \sum_{s=1}^{S} \nu_{1s}, ..., \sum_{s=1}^{S} \nu_{as}, ... \sum_{s=1}^{S} \nu_{As}) + g(x_{as}) + h(\boldsymbol{\nu}) \\ &= f(x_{as}) + g(x_{as}) + h(\boldsymbol{\nu}). \end{aligned} \tag{2.145}$$

So we have

$$u_{as}(x_{as}; \boldsymbol{\mu}; \boldsymbol{\nu}) = h(\boldsymbol{\nu}) \nu_{as} u_a(x_{as}; \boldsymbol{\mu}). \tag{2.146}$$

Equations (2.127) and (2.128) imply that

$$\frac{\partial^2 U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})}{\partial \nu_{at} \partial x_{as}} - \frac{\partial^2 U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})}{\partial \nu_{ar} \partial x_{as}} = c_{at} p_{as}, \tag{2.147}$$

$$\frac{\partial^2 U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})}{\partial \nu_{at} \partial x_{as}} - \frac{\partial^2 U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})}{\partial \nu_{ar'} \partial x_{as}} = c_{at} p_{as}. \tag{2.148}$$

Then we have

$$\frac{\partial^2 U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})}{\partial \nu_{ar} \partial x_{as}} - \frac{\partial^2 U(\boldsymbol{x}; \boldsymbol{\mu}; \boldsymbol{\nu})}{\partial \nu_{ar'} \partial x_{as}} = 0. \tag{2.149}$$

Due to equation (2.146), we have

$$\frac{\partial U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}} = \phi_a^{'}\left(h(\boldsymbol{\nu})\sum_{s=1}^{S}\nu_{as}u_a(x_{as};\boldsymbol{\mu});\boldsymbol{\mu};\boldsymbol{\nu}\right)h(\boldsymbol{\nu})\nu_{as}\frac{\partial u_{as}(x_{as};\boldsymbol{\mu})}{\partial x_{as}}. \tag{2.150}$$

Take derivative of equation (2.150) with respect to ν_{ar}, we have

$$\begin{aligned}
&\frac{\partial^2 U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \nu_{ar}\partial x_{as}} \\
&= \frac{\partial \phi_a^{'}\left(h(\boldsymbol{\nu})\sum_{s=1}^{S}\nu_{as}u_a(x_{as};\boldsymbol{\mu});\boldsymbol{\mu};\boldsymbol{\nu}\right)}{\partial \nu_{ar}}h(\boldsymbol{\nu})\nu_{as}\frac{\partial u_{as}(x_{as};\boldsymbol{\mu})}{\partial x_{as}} \\
&+ \phi_a^{''}\left(h(\boldsymbol{\nu})\sum_{s=1}^{S}\nu_{as}u_a(x_{as};\boldsymbol{\mu});\boldsymbol{\mu};\boldsymbol{\nu}\right)h(\boldsymbol{\nu})\nu_{as}\frac{\partial u_a(x_{as};\boldsymbol{\mu})}{\partial x_{as}}\frac{\partial h(\boldsymbol{\nu})}{\partial \nu_{ar}} \\
&\times \sum_{s=1}^{S}\nu_{as}u_a(x_{as};\boldsymbol{\mu}) \\
&+ \phi_a^{'}\left(h(\boldsymbol{\nu})\sum_{s=1}^{S}\nu_{as}u_a(x_{as};\boldsymbol{\mu});\boldsymbol{\mu};\boldsymbol{\nu}\right)\frac{\partial h(\boldsymbol{\nu})}{\partial \nu_{ar}}\nu_{as}\frac{\partial u_a(x_{as};\boldsymbol{\mu})}{\partial x_{as}}.
\end{aligned} \tag{2.151}$$

Equation (2.149) implies that

$$\frac{\partial h(\boldsymbol{\nu})}{\partial \nu_{ar}} = \frac{\partial h(\boldsymbol{\nu})}{\partial \nu_{ar'}}, \tag{2.152}$$

$$\frac{\partial \phi_a^{'}\left(h(\boldsymbol{\nu})\sum_{s=1}^{S}\nu_{as}u_a(x_{as};\boldsymbol{\mu});\boldsymbol{\mu};\boldsymbol{\nu}\right)}{\partial \nu_{ar}} = \frac{\partial \phi_a^{'}\left(h(\boldsymbol{\nu})\sum_{s=1}^{S}\nu_{as}u_a(x_{as};\boldsymbol{\mu});\boldsymbol{\mu};\boldsymbol{\nu}\right)}{\partial \nu_{ar'}}. \tag{2.153}$$

Then it means h is not function of $\boldsymbol{\nu}$, and ϕ_a is not function of $\boldsymbol{\nu}$. Then the objective function can be written as

$$\sum_{a=1}^{A}\phi_a\left(\sum_{s=1}^{S}\nu_{as}u_a(x_{as};\boldsymbol{\mu});\boldsymbol{\mu}\right). \tag{2.154}$$

Condition (2.111) uniquely determines the matrix $\boldsymbol{D}_{\boldsymbol{\mu}}\boldsymbol{x}(\boldsymbol{p}, I, \boldsymbol{\mu}, \boldsymbol{\nu})$. Since the matrix $\boldsymbol{\Sigma}$ has rank $AS-1$ and $\boldsymbol{\Sigma x} = \boldsymbol{0}$ only if $\boldsymbol{x} = c\boldsymbol{p}$ for some constant c, all solutions to equation $\boldsymbol{D}_{\boldsymbol{\mu}}\boldsymbol{x} = -\boldsymbol{K}\boldsymbol{D}^2_{\boldsymbol{\mu x}}U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})$ must have the form

$$
\boldsymbol{D}^2_{\boldsymbol{\mu x}} U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu}) = \begin{bmatrix} c_1 p_{11} + \lambda \frac{p_{11}}{\mu_1} & \dots & c_a p_{11} & \dots \\ c_1 p_{12} + \lambda \frac{p_{12}}{\mu_1} & \dots & c_a p_{12} & \dots \\ \dots & \dots & \dots & \dots \\ c_1 p_{1S} + \lambda \frac{p_{1S}}{\mu_1} & \dots & c_a p_{1S} & \dots \\ \dots & \dots & \dots & \dots \\ c_1 p_{a1} & \dots & c_a p_{a1} + \lambda \frac{p_{a1}}{\mu_a} & \dots \\ c_1 p_{a2} & \dots & c_a p_{a2} + \lambda \frac{p_{a2}}{\mu_a} & \dots \\ \dots & \dots & \dots & \dots \\ c_1 p_{aS} & \dots & c_a p_{aS} + \lambda \frac{p_{aS}}{\mu_a} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \tag{2.155}
$$

Therefore we have the following equations:

$$
\frac{\partial^2 U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \mu_{a'} \partial x_{as}} = c_{a'} p_{as} \text{ if } a^{'} \neq a, \tag{2.156}
$$

$$
\frac{\partial^2 U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \mu_{a'} \partial x_{as}} = c_a p_{as} + \lambda \frac{p_{as}}{\mu_a} \text{ if } a^{'} = a. \tag{2.157}
$$

The same trick like above will imply that

$$
\frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\mu_a}}{\partial \mu_{a'} \partial x_{as}} = \frac{c_{a'} p_{as}}{\mu_a}. \tag{2.158}
$$

Equation (2.158) implies

$$
\frac{\frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\mu_a}}{\partial \mu_{a'} \partial x_{as}}}{\frac{\partial^2 \frac{U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\mu_a}}{\partial \mu_{a'} \partial x_{ar}}} = \frac{\frac{\partial U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}}}{\frac{\partial U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{ar}}}. \tag{2.159}
$$

Since the objective function has the form (2.154), we have

$$\frac{\partial U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial x_{as}} = \phi_a^{'}\left(\sum_{s=1}^{S}\nu_{as}u_a(x_{as};\boldsymbol{\mu});\boldsymbol{\mu}\right)\nu_{as}\frac{\partial u_a(x_{as};\boldsymbol{\mu})}{\partial x_{as}}, \tag{2.160}$$

$$\begin{aligned}\frac{\partial^2 U(\boldsymbol{x};\boldsymbol{\mu};\boldsymbol{\nu})}{\partial \mu_{a'}\partial x_{as}} &= \frac{\partial \phi_a^{'}\left(\sum_{s=1}^{S}\nu_{as}u_a(x_{as};\boldsymbol{\mu});\boldsymbol{\mu}\right)}{\partial \mu_{a'}}\nu_{as}\frac{\partial u_a(x_{as};\boldsymbol{\mu})}{\partial x_{as}}\\ &+ \phi_a^{''}\left(\sum_{s=1}^{S}\nu_{as}u_a(x_{as};\boldsymbol{\mu});\boldsymbol{\mu}\right)\nu_{as}\frac{\partial u_a(x_{as};\boldsymbol{\mu})}{\partial x_{as}}\\ &\times \sum_{s=1}^{S}\nu_{as}\frac{\partial u_a(x_{as};\boldsymbol{\mu})}{\partial \mu_{a'}}\\ &+ \phi_a^{'}\left(\sum_{s=1}^{S}\nu_{as}u_a(x_{as};\boldsymbol{\mu});\boldsymbol{\mu}\right)\nu_{as}\frac{\partial^2 u_a(x_{as};\boldsymbol{\mu})}{\partial \mu_{a'}\partial x_{as}}.\end{aligned} \tag{2.161}$$

Equation (2.159) implies that

$$\frac{\frac{\partial^2 u_a(x_{as};\boldsymbol{\mu})}{\partial x_{as}\partial \mu_{a'}}}{\frac{\partial u_a(x_{as};\boldsymbol{\mu})}{\partial x_{as}}} = \frac{\frac{\partial^2 u_a(x_{ar};\boldsymbol{\mu})}{\partial x_{ar}\partial \mu_{a'}}}{\frac{\partial u_a(x_{ar};\boldsymbol{\mu})}{\partial x_{ar}}}. \tag{2.162}$$

Following the previous trick, we have

$$u_a(x_{as};\boldsymbol{\mu}) = f(\boldsymbol{\mu})u_a(x_{as}). \tag{2.163}$$

The objective function can be written as

$$\sum_{a=1}^{A}\phi_a\left(\sum_{s=1}^{S}\nu_{as}u_a(x_{as});\boldsymbol{\mu}\right). \tag{2.164}$$

Then the following will be obtained

$$\frac{\partial\frac{\phi_a}{\mu_a}}{\partial x_{as}} = \frac{\partial\frac{\phi_a\left(\sum_{s=1}^{S}\nu_{as}u_a(x_{as});\boldsymbol{\mu}\right)}{\mu_a}}{\partial u}\nu_{as}\frac{\partial u_a(x_{as})}{\partial x_{as}}, \tag{2.165}$$

$$\frac{\partial^2\frac{\phi_a}{\mu_a}}{\partial \mu_{a'}\partial x_{as}} = \frac{\partial^2\frac{\phi_a\left(\sum_{s=1}^{S}\nu_{as}u_a(x_{as});\boldsymbol{\mu}\right)}{\mu_a}}{\partial \mu_{a'}\partial u}\nu_{as}\frac{\partial u_a(x_{as})}{\partial x_{as}}, \tag{2.166}$$

$$\frac{\partial^2\frac{\phi_i}{\mu_i}}{\partial \mu_{a'}\partial x_{is}} = \frac{\partial^2\frac{\phi_i\left(\sum_{s=1}^{S}\nu_{is}u_i(x_{is});\boldsymbol{\mu}\right)}{\mu_i}}{\partial \mu_{a'}\partial u}\nu_{is}\frac{\partial u_i(x_{is})}{\partial x_{is}}. \tag{2.167}$$

Then the following holds

$$\frac{\frac{\partial^2 \frac{\phi_a}{\mu_a}}{\partial \mu_{a'} \partial x_{as}}}{\frac{\partial^2 \frac{\phi_i}{\mu_i}}{\partial \mu_{a'} \partial x_{is}}} = \frac{\frac{\partial \frac{\phi_a}{\mu_a}}{\partial x_{as}}}{\frac{\partial \frac{\phi_i}{\mu_i}}{\partial x_{is}}}. \tag{2.168}$$

Then by the same trick, we can show that

$$\phi_a \left(\sum_{s=1}^{S} \nu_{as} u_a(x_{as}); \boldsymbol{\mu} \right) = \mu_a \phi \left(\sum_{s=1}^{S} \nu_{as} u_a(x_{as}) \right). \tag{2.169}$$

Then it can be shown that $u_a(x) = u_{a'}(x) = u(x)$.

□

Chapter 3

The identification of smooth ambiguity preference

3.1 Introduction

The question in this chapter is: if the observed asset demand functions are generated by some ambiguity preference, can the underlying ambiguity preference be uniquely recovered? In Chapter 2, we explicitly construct ambiguity preferences which rationalize the observations if the observations pass the revealed preference test. However, such construction is not unique, since finite observations cannot trace out individual indifference curves. To uniquely identify underlying ambiguity preference in incomplete markets, we assume the demand functions are given. Suppose the observations from demand functions pass the revealed preference tests, we establish the conditions under which we can uniquely recover the ambiguity preference.

Before we proceed to the identification of ambiguity preference, let's first revisit the identification of risk preference in Green, Lau, and Polemarchakis (1979) and Dybvig and Polemarchakis (1981). We write the optimization problem in its general form to incorporate the pure risk case: suppose the individual makes portfolio choice, at prices of assets $\boldsymbol{p}$, the optimization problem of the individual is

$$\max_{y \in \mathbb{R}^J} U(\boldsymbol{R}\boldsymbol{y})$$

$$s.t. \quad \boldsymbol{p}\boldsymbol{y} = 1.$$

Under strict quasi-concavity assumption and Assumption 2, a solution to the optimization problem, $\boldsymbol{y}(\boldsymbol{p})$, exists and is unique; it defines the demand function for assets,

$$\boldsymbol{y} : \boldsymbol{P} \to \mathbb{R}^J.$$

There exists an open set of prices, $\boldsymbol{P}^0 \subset \boldsymbol{P}$, for which the solution to the optimization problem satisfies $\boldsymbol{y}(\boldsymbol{p}) \in \boldsymbol{Y}$. Importantly, the demand function is invertible: given $\boldsymbol{y} \in \boldsymbol{Y}$, there is a unique $\boldsymbol{p}(\boldsymbol{y}) \in \boldsymbol{P}^0$, such that $\boldsymbol{y} = \boldsymbol{y}\big(\boldsymbol{p}(\boldsymbol{y})\big)$. We restrict attention to the demand function for assets

$$\boldsymbol{y} : \boldsymbol{P}^0 \to \boldsymbol{Y}.$$

The demand for assets satisfies necessary and sufficient first order conditions

$$\boldsymbol{D}U(\boldsymbol{R}\boldsymbol{y}) = \lambda \boldsymbol{p}, \lambda > 0;$$

$$\boldsymbol{p}\boldsymbol{y} = 1.$$

As a consequence, it identifies the family of marginal rate of substitution of assets functions

$$m_{jk} : \boldsymbol{Y} \to (0, \infty)$$

defined by

$$m_{jk}(\boldsymbol{y}) = \frac{\frac{\partial U(\boldsymbol{R}\boldsymbol{y})}{\partial y_j}}{\frac{\partial U(\boldsymbol{R}\boldsymbol{y})}{\partial y_k}}.$$

3.2 Identification under pure risk

The probability measure over states of risk is

$$\boldsymbol{\nu} \in \boldsymbol{\Delta}(\boldsymbol{S}),$$

and the utility function of the individual is

$$U = E_{\boldsymbol{\nu}} u.$$

Assumption 3

(1) u is analytic on $\mathbb{R}_+$, is strictly increasing and strictly concave;

(2) $prob\{\boldsymbol{r}_j \geq 0\}$ =1, $prob\{\boldsymbol{r}_j = 0\} \neq 1$, $\boldsymbol{r}_j$ is linearly independent with return vectors of other assets, and $E\boldsymbol{r}_j^l < +\infty$ for all natural number l.

Assumption $3^{'}$

(1) u is $\boldsymbol{C}^2$ on $\mathbb{R}_{++}$, is strictly increasing and strictly concave;

(2) $prob\{\boldsymbol{r}_j \geq 0\}$ =1, $prob\{\boldsymbol{r}_j = 0\} \neq 1$, $\boldsymbol{r}_j$ is linearly independent with return vectors of other assets, and $E\boldsymbol{r}_j^l < +\infty$ for $l = 1, 2$.

Assumption $3^{''}$

(1) u is $\boldsymbol{C}^\infty$ on $\mathbb{R}_{++}$, is strictly increasing and strictly concave, and $u^{(n)} = \frac{\partial^n u}{\partial x^n} \neq 0$, at some $\overline{x} \in (0, \infty)$, $n = 1, ...$;

(2) $prob\{\boldsymbol{r}_j \geq 0\}$ =1, $prob\{\boldsymbol{r}_j = 0\} \neq 1$, $\boldsymbol{r}_j$ is linearly independent with return vectors of other assets, and $E\boldsymbol{r}_j^l < +\infty$ for $l = 1, 2$.

At prices of assets $\boldsymbol{p} \in \boldsymbol{P}^o$, the optimization problem of the individual is

$$\max_{\boldsymbol{y} \in \mathbb{R}^J} E_{\boldsymbol{\nu}} u(\boldsymbol{R}\boldsymbol{y})$$
$$s.t. \quad \boldsymbol{p}\boldsymbol{y} = 1.$$

The demands for assets satisfies necessary and sufficient first order conditions

$$E_{\boldsymbol{\nu}} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{R} = \lambda \boldsymbol{p}, \lambda > 0;$$

$$\boldsymbol{p}\boldsymbol{y} = 1.$$

As a consequence, the demand for assets identifies the family of marginal rates of substitution

$$m_{jk}(\boldsymbol{y}) = \frac{E_{\boldsymbol{\nu}} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j}{E_{\boldsymbol{\nu}} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k} > 0.$$

Lemma 2 (Green, Lau and Polemarchakis, 1979)**.** *Suppose that*

(1) the utility index u, and asset return $\boldsymbol{r}_j$ for $j = 1, ..., J$ satisfy Assumption 3;

(2) the probability measure over states of risk, $\boldsymbol{\nu} \in \boldsymbol{\Delta}(\boldsymbol{S})$, is known.

Then, the demand for assets identifies the cardinal index for risk u up to a positive affine transformation.

Proof. Without loss of generality, we normalize $u(0) = 0$, and $u^{'}(0) = 1$.

Differentiation of the functional equation

$$E_{\boldsymbol{\nu}} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j = m_{jk}(\boldsymbol{y}) E_{\boldsymbol{\nu}} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k, \tag{3.1}$$

with respect to y_k yields

$$E_{\boldsymbol{\nu}} u^{''}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j\boldsymbol{r}_k = \frac{\partial m_{jk}(\boldsymbol{y})}{\partial y_k} E_{\boldsymbol{\nu}} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k + m_{jk}(\boldsymbol{y}) E_{\boldsymbol{\nu}} u^{''}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k^2. \tag{3.2}$$

At $\boldsymbol{y}$=(0,...,0), we have

$$u^{''}(0)(E\boldsymbol{r}_j\boldsymbol{r}_k - E\boldsymbol{r}_k^2) = \frac{\partial m_{jk}(\boldsymbol{y})}{\partial y_k} E_{\boldsymbol{\nu}}\boldsymbol{r}_k. \tag{3.3}$$

Repeat the above differentiation with respect to y_j, and also evaluate at $\boldsymbol{y} = (0, ..., 0)$, we have

$$u^{''}(0)(E\boldsymbol{r}_j\boldsymbol{r}_k - E\boldsymbol{r}_j^2) = -\frac{\partial m_{jk}(\boldsymbol{y})}{\partial y_j} E_{\boldsymbol{\nu}}\boldsymbol{r}_k. \tag{3.4}$$

Hold's inequality can be used to show that $E\boldsymbol{r}_j\boldsymbol{r}_k - E\boldsymbol{r}_k^2 = 0$ and $E\boldsymbol{r}_j\boldsymbol{r}_k - E\boldsymbol{r}_j^2 = 0$ cannot hold at the same time, so we can uniquely recover $u^{''}(0)$. Higher-order differentiations will recover higher-order derivatives of u at 0, which will recover index u given the assumptions on u. □

Remark 10. Green, Lau, and Polemarchakis (1979) do not require a risk-free asset. Instead, the cardinal risk index is analytic at $x = 0$.

Lemma 3 (Dybvig and Polemarchakis, 1981)**.** *Suppose that*

(1) the utility index u, and asset return $\boldsymbol{r}_j$ for $j = 1, ..., J$ satisfy Assumption $3^{'}$;

(2) there is an asset that is risk-free: $\boldsymbol{r}_1 = 1$ across states of risk;

(3) the probability measure over states of risk, $\boldsymbol{\nu} \in \boldsymbol{\Delta}(\boldsymbol{S})$, is known.

Then, the demand for assets identifies the cardinal index for risk u up to a positive affine transformation.

Proof. Differentiation of the functional equation

$$E_{\boldsymbol{\nu}} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j = m_{jk}(\boldsymbol{y}) E_{\boldsymbol{\nu}} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k, \tag{3.5}$$

with respect to y_k yields

$$E_{\boldsymbol{\nu}} u^{''}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j\boldsymbol{r}_k = \frac{\partial m_{jk}(\boldsymbol{y})}{\partial y_k} E_{\boldsymbol{\nu}} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k + m_{jk}(\boldsymbol{y}) E_{\boldsymbol{\nu}} u^{''}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k^2. \tag{3.6}$$

For $j = 1$,

$$E_{\boldsymbol{\nu}} u^{''}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k = \frac{\partial m_{jk}(\boldsymbol{y})}{\partial y_k} E_{\boldsymbol{\nu}} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k + m_{jk}(\boldsymbol{y}) E_{\boldsymbol{\nu}} u^{''}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k^2, \tag{3.7}$$

which, evaluated at the portfolio $\boldsymbol{y} = (x, 0, ..., 0)$, yields

$$u^{''}(x) = \frac{\partial m_{jk}(x, 0, ..., 0)}{\partial y_k} u^{'}(x) + m_{jk}(x, 0, ..., 0) u^{''}(x) E_{\boldsymbol{\nu}} \boldsymbol{r}_k^2. \tag{3.8}$$

Since $m_{jk}(x, 0, ..., 0)u^{''}(x) < 0$, this identifies the risk-aversion of the individual,

$$-\frac{u^{''}(x)}{u^{'}(x)} = \frac{\frac{\partial m_{jk}(x,0,...,0)}{\partial y_k}}{m_{jk}(x, 0, ..., 0) E_{\boldsymbol{\nu}} \boldsymbol{r}_k^2 - 1}, x \in (0, \infty), \tag{3.9}$$

or, equivalently, the cardinal risk index, u, up to an affine transformation. To see this, notice that the right hand side is observable, and we integrate both sides to get $\ln^{u^{'}(x)} + \ln^{B}$, where B is an integration constant. Exponentiate and integrate again, we get $Bu(x) + C$,where C is an integration constant, which is positive affine transformation of $u(x)$. □

Remark 11. The argument does not require full knowledge of the distribution of payoff, $(\boldsymbol{R}, \boldsymbol{\nu})$, only of the second moment of the payoff of a risky asset.

Lemma 4 (Polemarchakis,1983)**.** *Suppose that*

(1) the utility index u, and asset return $\boldsymbol{r}_j$ for $j = 1, ..., J$ satisfy Assumption $3^{''}$;

(2) there is an asset that is risk-free: $\boldsymbol{r}_1 = 1$ across states of risk;

(3) the cardinal risk index u is known.

Then, the demand for assets identifies all moments of the distribution of payoffs of assets.

Proof. Without loss of generality, u is smooth and $u^n(1) \neq 0$ at $\overline{x} = 1$. As previously, only, here, with $u^{(1)}(x) = u^{'}(x)$ and $u^{(2)}(x) = u^{''}(x)$,

$$E_{\boldsymbol{\nu}} u^{(2)}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j\boldsymbol{r}_k = \frac{\partial m_{jk}(\boldsymbol{y})}{\partial y_k} E_{\boldsymbol{\nu}} u^{(1)}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k + m_{jk}(\boldsymbol{y}) E_{\boldsymbol{\nu}} u^{(2)}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_k^2, \quad (3.10)$$

which, evaluated at $\boldsymbol{y} = (1, 0, ..., 0)$, and with $u^{(1)} = 1$, yields

$$E_{\boldsymbol{\nu}} u^{(2)}(1)\boldsymbol{r}_j\boldsymbol{r}_k = \frac{\partial m_{jk}(\boldsymbol{y})}{\partial y_k} + m_{jk}(\boldsymbol{y}) u^{(2)}(1) E_{\boldsymbol{\nu}} \boldsymbol{r}_k^2. \quad (3.11)$$

Since $m_{jk}(\boldsymbol{y}) > 0$, while $u^2(1) \neq 0$, this identifies

$$E_{\boldsymbol{\nu}} \boldsymbol{r}_k^2 \quad \text{and} \quad E_{\boldsymbol{\nu}} \boldsymbol{r}_j \boldsymbol{r}_k,$$

the second moments of the distribution of payoffs of assets. Higher order derivatives identify higher moments; at each step the coefficients of the moments to be identified do not vanish. □

Remark 12. Instead of the cardinal risk index, u, it suffices to know $E_{\boldsymbol{\nu}} \boldsymbol{r}_k^2$, the second moment of the distribution of returns of a risky asset.

Remark 13. Knowing second moment of the return of one risky asset is necessary for above recovery argument. One example is this: an investor with a CARA cardinal utility index, $u(x) = -e^{-\rho x}$ demands a risky asset with normally distributed payoffs, $\boldsymbol{r}_2 \sim N(\mu, \sigma^2)$ against a risk-free asset with payoff $\boldsymbol{r}_1 = r$ according to $y_2 = \frac{\mu - r}{\rho \sigma^2}$; it follows that simultaneous identification of the cardinal risk aversion index and the distribution of payoffs assets, without any, even partial, knowledge of either, is not possible.

3.3 Identification under uncertainty

The family of conditional probability measures over states of risk is

$$\boldsymbol{\nu} : \boldsymbol{A} \rightarrow \boldsymbol{\Delta}(\boldsymbol{S}),$$

and the utility function of the individual is

$$U = \Phi(..., w_a, ...),$$

where u is a cardinal risk index, $w_a = u^{-1}(E_{\boldsymbol{\nu}_a} u)$ is the certainty equivalent of the distribution of wealth at a state of ambiguity, and Φ is an ordinal utility function over the distribution of certainty equivalent wealth across states of uncertainty. The utility function satisfies Assumption 1.

At prices of assets $\boldsymbol{p} \in \boldsymbol{P}^o$, the optimization problem of individual is

$$\max_{\boldsymbol{y} \in \mathbb{R}^J} U = \Phi(..., w_a, ...)$$

$$s.t. \; w_a(\boldsymbol{Ry}) = u^{-1}\big(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})\big),$$

$$\boldsymbol{py} = 1.$$

The demand for assets satisfies necessary and sufficient first order conditions

$$\sum_a \frac{\partial \Phi}{\partial w_a} \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{Ry})\boldsymbol{R}}{u^{'}\left(u^{-1}\big(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})\big)\right)} = \lambda \boldsymbol{p}, \; \lambda > 0,$$

$$\boldsymbol{py} = 1.$$

As a consequence, the demand for assets identifies the family of marginal rates of substitution

$$m_{jk}(\boldsymbol{y}) = \frac{\sum_a \frac{\partial \Phi}{\partial w_a} \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{Ry})\boldsymbol{r}_j}{u^{'}\left(u^{-1}\left(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})\right)\right)}}{\sum_a \frac{\partial \Phi}{\partial w_a} \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{Ry})\boldsymbol{r}_k}{u^{'}\left(u^{-1}\left(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})\right)\right)}} > 0.$$

3.3.1 Identification with an ambiguity-free asset

The probability distribution $\boldsymbol{\nu}_a$ conditional on each ambiguity state a will generate a distribution of return for each asset. We say one asset is *ambiguity-free* if the its conditional distributions are equal across ambiguity states.

Definition 3. *Ambiguity-free asset*
An asset is *ambiguity-free* if its return distributions conditional on each ambiguity state are the same across states of ambiguity.

Example 1. Suppose there are 3 states of risk, i.e. $S = 3$, and 2 states of uncertainty, i.e. $A = 2$. Assume the asset pays $(1, a, a)$ contingent on risk states. The probability distributions conditional on two uncertainty states are $(\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, 0)$. Then this asset is ambiguity-free.

Since the existence of an ambiguity-free asset is important for our identification argument, it deserves a bit more discussion. Does there exist an ambiguity-free asset given any conditional probabilities? The risk-free asset is one, but it's trivial.[1] The existence of a risky ambiguity-free asset is not guaranteed for arbitrary conditional probabilities. The reasoning is this: if A conditional probabilities over S-dimension return vector generate the same return distribution, then they must have the same mean; however, if $A > S$, the existence of such S-dimension return vector is not generic.

The existence of a risky ambiguity-free asset relies on underlying conditional probabilities. If we restrict the space of conditional probabilities, the existence of such asset is not a problem. One restriction which generates an ambiguity-free asset is that we focus on such probability space: $\mathfrak{P} = \{\boldsymbol{\nu}_a : \nu_{11} = ... = \nu_{a1} = ... = \nu_{A1}\}$, that is, all conditional probabilities in this probability space put the same probability on the first state. Then any return vector $(a, b, ..., b)$ is ambiguity-free and risky for $a \neq b$. The restricted probability space $\mathfrak{P}$ is not generic in (non-restricted) probability space, however, such space is big enough for us to work on, and it is widely used in experimental work. For example, in Ahn, Choi, Gale, and Kariv (2014), the composition of one color balls is publicly announced, and the composition of other two color balls is unknown. This falls in our setting, and one ambiguity-free asset can be traded.

Proposition 4. *Suppose that*

(1) the objective function satisfies Assumption 1, and asset return satisfies Assumption 2;

[1] In the remaining of this chapter, when we refer to an ambiguity-free asset, we will implicitly mean it is ambiguity-free and risky, even though we do not explicitly emphasize its riskiness property.

(2) there is an asset $j = 1$ that is risk-free: $\boldsymbol{r}_1 = 1$ across states of the world;

(3) there is an asset $j = 2$ that is ambiguity-free: its payoff distribution is invariant to the states of ambiguity;

(4) the family of conditional probability measures over states of risk, $\boldsymbol{\nu} : \boldsymbol{A} \to \boldsymbol{\Delta(S)}$ is known;

(5) the matrix of expected marginal utility, $\left[..., \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{Ry})\boldsymbol{R}}{u^{'}\left(u^{-1}\left(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})\right)\right)}, ... \right]$ has full row rank A at each portfolio $\boldsymbol{y}$.

Then, the demand for assets identifies the cardinal index for risk, u, up to a positive affine transformation, as well as the ordinal utility function Φ, up to a monotonically increasing transformation.

Proof. Step 1—recovering risk index:

Restrict attention to portfolios $\boldsymbol{y} = (y_1, y_2, 0, ..., 0)$, and let $\tilde{\boldsymbol{y}} = (y_1, y_2)$ be the associated truncated portfolio.

Since the distribution of payoffs of assets 1 and 2 are invariant across states of ambiguity, there exists a probability measure, $\tilde{\boldsymbol{\nu}} \in \boldsymbol{\Delta(S)}$, and a matrix of payoffs of assets over states of risk $\tilde{\boldsymbol{R}} = (\boldsymbol{1}_{\#\boldsymbol{S}}, \tilde{\boldsymbol{r}_2})$, such that, the distribution of payoffs of assets generated by $(\boldsymbol{\nu}_a, \boldsymbol{Ry})$, for any state of ambiguity, coincides with the distribution generated by $(\tilde{\boldsymbol{\nu}}, \tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})$.

As a consequence,

$$m_{12}(\tilde{\boldsymbol{y}}) = \frac{E_{\tilde{\boldsymbol{\nu}}} u^{'}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})}{E_{\tilde{\boldsymbol{\nu}}} u^{'}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})\tilde{\boldsymbol{r}_2}} > 0. \tag{3.12}$$

Identification of the cardinal risk index, u, then follows as in Lemma 3.

Step 2—recovering uncertainty index:

The first order conditions for an optimum,

$$\sum_a \frac{\partial \Phi}{\partial w_a} \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{R}}{u^{'}\left(u^{-1}\big(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})\big)\right)} = \lambda \boldsymbol{p},\ \ \lambda > 0, \tag{3.13}$$

can be written in matrix form,

$$[\Phi_1, ..., \Phi_a, ...\Phi_A] \begin{bmatrix} & & \\ & c_{aj} & \\ & & \end{bmatrix} = [\lambda p_1, ..., \lambda p_j, ...\lambda p_J], \tag{3.14}$$

where $\Phi_a = \frac{\partial \Phi}{\partial w_a}$, $c_{aj} = \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{R}\boldsymbol{y})\boldsymbol{r}_j}{u^{'}\left(u^{-1}\left(E_{\boldsymbol{\nu}_a} u(\boldsymbol{R}\boldsymbol{y})\right)\right)}$, and matrix $\boldsymbol{C}$ has dimension A times J.

Since we have recovered index u and conditional distribution of asset return is known, the matrix $\boldsymbol{C}$ is computable. If matrix $\boldsymbol{C}$ has full row rank, then

$$[\Phi_1, ..., \Phi_a, ...\Phi_A] = [\lambda p_1, ..., \lambda p_j, ...\lambda p_J]\boldsymbol{C}^T[\boldsymbol{C}\boldsymbol{C}^T]^{-1}. \tag{3.15}$$

So we can trace out the marginal rate of substitution $\frac{\Phi_a}{\Phi_a^{'}}$. Under assumption 1, Φ is strictly quasi-concave, continuously differentiable, and has strictly positive gradient everywhere on $\mathbb{R}_+^A$, following Mas-Colell (1977), knowing the marginal rate of substitution $\frac{\Phi_a}{\Phi_a^{'}}$ will identify function Φ up to monotonically increasing transformation. $\square$

In Klibanoff, Marinacci, and Mukerji (2005), the smooth ambiguity utility functional form is $U = E_{\boldsymbol{\mu}}\phi\Bigg(u^{-1}\big(E_{\boldsymbol{\nu}_a} u(x_s)\big)\Bigg)$, which is a special case of the above utility functional form, so the identification follows directly from Proposition 4.

Corollary 2. *Suppose that*

(1) the objective function satisfies Assumption $1^{'}$*, and asset return satisfies Assumption 2;*

(2) there is an asset $j = 1$ *that is risk-free:* $\boldsymbol{r}_1 = 1$ *across states of the world;*

(3) there is an asset $j = 2$ that is ambiguity-free: its payoff distribution is invariant to the states of ambiguity;

(4) the family of conditional probability measures over states of risk, $\boldsymbol{\nu} : \boldsymbol{A} \to \boldsymbol{\Delta(S)}$ is known;

(5) the payoffs $E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} (\boldsymbol{r}_3)^2$ and $E_{\boldsymbol{\mu}} (E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3)^2$ of an ambiguous asset $j = 3$ are known and satisfy $(E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3)^2 \neq E_{\boldsymbol{\mu}} (E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3)^2$.

Then, the demand for assets identifies the risk index u and uncertainty index ϕ up to positive affine transformation.

Proof. The identification of index u follows the same argument in Proposition 4, we sketch the recovery of index ϕ.

The marginal rate of substitution between risk-free asset 1 and ambiguous asset 3 gives

$$E_{\boldsymbol{\mu}} \phi^{'} \left(u^{-1} \big(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry}) \big) \right) \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{Ry}) \boldsymbol{r}_1}{u^{'} \left(u^{-1} \big(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry}) \big) \right)} =$$
$$m_{13}(\boldsymbol{y}) E_{\boldsymbol{\mu}} \phi^{'} \left(u^{-1} \big(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry}) \big) \right) \frac{E_{\boldsymbol{\nu}_a} u^{'}(\boldsymbol{Ry}) \boldsymbol{r}_3}{u^{'} \left(u^{-1} \big(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry}) \big) \right)}. \tag{3.16}$$

Take derivative on both sides with respect to y_3, and evaluate at $\boldsymbol{y} = (x, 0, ..., 0)$, we get

$$[(E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3)^2 - E_{\boldsymbol{\mu}} (E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3)^2] \frac{\phi^{''}(x)}{\phi^{'}(x)} =$$
$$[E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} (\boldsymbol{r}_3)^2 - E_{\boldsymbol{\mu}} (E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3)^2] \frac{u^{''}(x)}{u^{'}(x)} + (E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3)^2 \frac{\partial m_{13}(x, 0, ..., 0)}{\partial y_3}. \tag{3.17}$$

Given risk index u recovered, this will identify uncertainty index ϕ uniquely up to a positive affine transformation. □

Remark 14. The coefficient of $\frac{\phi^{''}(x)}{\phi^{'}(x)}$ is the variance of random variable $E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3$ evaluated by ambiguity probability measure $\boldsymbol{\mu}$, and nonzero of the coefficient requires ambiguity on the mean. The full row rank condition implies the non-vanishing of the coefficient, but this condition is much weaker than full row rank condition.

Remark 15. The above recovery argument entails knowing some moments of assets evaluated by both $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$: $E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3$, $E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} (\boldsymbol{r}_3)^2$, and $E_{\boldsymbol{\mu}} (E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3)^2$. If the domain of preference is compound *objective* lotteries, such moments can be computed directly from the known objective probabilities. However, if the domain is subjective-objective two stage lotteries within Anscombe and Aumann (1963) framework, these payoff moments are not directly observable, and should be elicited from subjects.

In the above recovery arguments, we only observe individual portfolio choice; however, in reality, individual makes joint decision on consumption and portfolio. In this case, individual optimization problem can be written as

$$\max_{\boldsymbol{y} \in \mathbb{R}^J} U = \Phi\Big(c_0, \phi^{-1}\big(E_{\boldsymbol{\mu}} \phi(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})))\big)\Big)$$
$$s.t. \; p_0 c_0 + \boldsymbol{py} = 1.$$

Assumption $1''$

(1) u is $\boldsymbol{C}^2$ on $\mathbb{R}_{++}$, is strictly concave and satisfies $\forall x \in \mathbb{R}_{++}$, $u' > 0$;

(2) Φ is $\boldsymbol{C}^1$ on $\mathbb{R}^2_{++}$, is strictly concave, and satisfies $\forall \boldsymbol{c} \in \mathbb{R}^2_{++}$, $\Phi_i > 0$ for all $i = 1, 2$;

(3) $\phi^{-1}\big(E_{\boldsymbol{\mu}} \phi(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{x})))\big)$ is strictly concave on $\mathbb{R}_{++}$.

The demand for assets satisfies necessary and sufficient first order conditions

$$\Phi_1 = \lambda p_0, \tag{3.18}$$

$$\Phi_2 \frac{E_{\boldsymbol{\mu}} \phi'\left(u^{-1}\big(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})\big)\right) \frac{E_{\boldsymbol{\nu}_a} u'(\boldsymbol{Ry}) \boldsymbol{R}}{u'(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})))}}{\phi'\left(\phi^{-1}\big(E_{\boldsymbol{\mu}} \phi(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})))\big)\right)} = \lambda \boldsymbol{p}, \;\; \lambda > 0, \tag{3.19}$$

$$p_0 c_0 + \boldsymbol{py} = 1. \tag{3.20}$$

Corollary 3. *Suppose that*

(1) the objective function satisfies Assumption $1''$, and asset return satisfies Assumption 2;

(2) *there is an asset $j = 1$ that is risk-free: $\boldsymbol{r}_1 = 1$ across states of the world;*

(3) *there is an asset $j = 2$ that is ambiguity-free: its payoff distribution is invariant to the states of ambiguity;*

(4) *the family of conditional probability measures over states of risk, $\boldsymbol{\nu} : \boldsymbol{A} \to \boldsymbol{\Delta(S)}$ is known;*

(5) *the payoffs $E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a}(\boldsymbol{r}_3)^2$ and $E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3)^2$ of an ambiguous asset $j = 3$ are known and satisfy $(E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3)^2 \neq E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3)^2$.*

Then, the demand for consumption and assets identifies the cardinal index for risk, u, as well as the cardinal index for uncertainty ϕ, up to a positive affine transformation, the ordinal index for time preference, Φ, up to a monotonically increasing transformation.

Proof. Step 1−recovering risk index:

First order conditions trace out the marginal rate of substitution between asset 1 and asset 2,

$$\frac{E_{\boldsymbol{\mu}}\phi'\left(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\big)\right)\frac{E_{\boldsymbol{\nu}_a}u'(\boldsymbol{Ry})\boldsymbol{r}_1}{u'\left(u^{-1}\left(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\right)\right)}}{E_{\boldsymbol{\mu}}\phi'\left(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\big)\right)\frac{E_{\boldsymbol{\nu}_a}u'(\boldsymbol{Ry})\boldsymbol{r}_2}{u'\left(u^{-1}\left(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\right)\right)}} = m_{12}(c_0, \boldsymbol{y}). \tag{3.21}$$

Since the return distribution of asset 1 and 2 is ambiguity free, at $\tilde{\boldsymbol{y}} = (y_1, y_2, 0, ..., 0)$, we have

$$m_{12}(c_0, \tilde{\boldsymbol{y}}) = \frac{E_{\tilde{\boldsymbol{\nu}}}u'(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})}{E_{\tilde{\boldsymbol{\nu}}}u'(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})\tilde{\boldsymbol{r}}_2} > 0. \tag{3.22}$$

Then recovery of u follows from Lemma 3.

Step 2−recovering uncertainty index:

The marginal rate of substitution between risk-free asset 1 and ambiguous asset 3 gives

$$E_{\boldsymbol{\mu}}\phi'\left(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\big)\right)\frac{E_{\boldsymbol{\nu}_a}u'(\boldsymbol{Ry})\boldsymbol{r}_1}{u'\left(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\big)\right)} =$$

$$m_{13}(c_0,\boldsymbol{y})E_{\boldsymbol{\mu}}\phi'\left(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\big)\right)\frac{E_{\boldsymbol{\nu}_a}u'(\boldsymbol{Ry})\boldsymbol{r}_3}{u'\left(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\big)\right)}. \tag{3.23}$$

Take derivative on both sides with respect to y_3, and evaluate at $\boldsymbol{y} = (x, 0, ..., 0)$, we get

$$[(E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3)^2 - E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3)^2]\frac{\phi''(x)}{\phi'(x)} =$$

$$[E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}(\boldsymbol{r}_3)^2 - E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3)^2]\frac{u''(x)}{u'(x)} + (E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3)^2\frac{\partial m_{13}(c_0, x, 0, ..., 0)}{\partial y_3}. \tag{3.24}$$

Given risk index u recovered, this will identify uncertainty index ϕ uniquely up to a positive affine transformation.

Step 3—recovering time index:

The marginal rate of substitution between risk-free asset 1 and consumption c_0 gives

$$\frac{\Phi_1}{\Phi_2 E_{\boldsymbol{\mu}}\phi'\left(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\big)\right)\frac{E_{\boldsymbol{\nu}_a}u'(\boldsymbol{Ry})\boldsymbol{r}_2}{u'\left(u^{-1}\left(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\right)\right)}} = m_{01}(c_0, \boldsymbol{y}). \tag{3.25}$$

Evaluate at $\boldsymbol{y} = (x, 0, ..., 0)$, we have

$$\frac{\Phi_1(c_0, x)}{\Phi_2(c_0, x)} = m_{01}(c_0, x, 0, ..., 0)E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_1. \tag{3.26}$$

It will recover Φ up to monotonically increasing transformation. □

Remark 16. The functional form $\Phi\Big(c_0, \phi^{-1}\big(E_{\boldsymbol{\mu}}\phi(u^{-1}(E_{\boldsymbol{\nu}_a}u(x_s)))\big)\Big)$ contains one important special case where the three preference parameters—elasticity of inter-temporal substitution, risk aversion and ambiguity aversion are separated as in Hayashi and Miao (2011), which shows that such separation will explain the historical data better.

Remark 17. As in Lemma 3 and Lemma 4, knowledge of the second moment of the distribution of payoffs of an asset invariant across states of ambiguity permits identification of the risk index u, as well as identification of the payoffs of assets invariant over states of ambiguity.

Example 2. *Identification of risk and uncertainty aversion*
Suppose there are one riskless asset with payoff $\boldsymbol{r}$, one ambiguity free asset with payoff $\boldsymbol{r}_1$, $\boldsymbol{r}_1 \sim N(\mu_1, \sigma_1^2)$ and one ambiguous asset with payoff $\boldsymbol{r}_2$, $\boldsymbol{r}_2 \sim N(\mu_2, \sigma_2^2)$, where individual has ambiguity on the mean of $\boldsymbol{r}_2$, and $\boldsymbol{\mu}_2 \sim N(\theta, \sigma_0^2)$. It is assumed that payoffs of assets are independent. Individual is endowed with risk preference $u(x) = -\frac{e^{-\rho x}}{\rho}$, and ambiguity preference $\phi = -\frac{e^{-Au}}{A}$. Individual will demand the ambiguity free asset $\alpha_1 = \frac{\mu_1 - r}{\rho\sigma_1^2}$, and the ambiguous asset $\alpha_2 = \frac{\theta - r}{\rho\sigma_2^2 + A\sigma_0^2}$. It follows that risk aversion index u can be recovered from ambiguity-free asset demand α_1 if we know its return distribution; and ambiguity aversion index ϕ can be recovered from ambiguous asset demand α_2 if we know its conditional distribution and ambiguity.

Remark 18. The above argument identifies cardinal risk index u and uncertainty index ϕ once knowing certain moments of asset returns under ambiguity probability measure $\boldsymbol{\mu}$. One question is: suppose we know conditional distribution of asset return, can we recover without reference to ambiguity probability measure $\boldsymbol{\mu}$? The above example shows that this is not the case: even we know payoffs of the riskless asset $\boldsymbol{r}$, and payoffs of the ambiguity-free risky asset μ_1, σ_1, and all conditional distribution of ambiguous asset θ, σ_2, we can not identify risk index u and uncertainty index A uniquely.

The above identification argument requires observing individual asset demand. An equivalent way to establish recoverability of risk and uncertainty preference is to know individual portfolio indifference correspondence

$$I(\boldsymbol{y}) = \left\{ \boldsymbol{x} \in \mathbb{R}^J : E_{\boldsymbol{\mu}}\phi\Big(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{R}\boldsymbol{x})\big)\Big) = E_{\boldsymbol{\mu}}\phi\Big(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{R}\boldsymbol{y})\big)\Big) \right\}.[2]$$

Under the same assumption as above on asset return and individual

[2] We illustrate the argument using the this special functional form, but the argument here can recover more general utility function.

belief, risk index u and uncertainty index ϕ can be identified from individual portfolio indifference correspondence.

Corollary 4. *Suppose that*

(1) the objective function satisfies Assumption $1'$*, and asset return satisfies Assumption 2;*

(2) there is an asset $j = 1$ *that is risk-free:* $\boldsymbol{r}_1 = 1$ *across states of the world;*

(3) there is an asset $j = 2$ *that is ambiguity-free: its payoff distribution is invariant to the states of ambiguity;*

(4) the family of conditional probability measures over states of risk, $\boldsymbol{\nu} : \boldsymbol{A} \to \boldsymbol{\Delta(S)}$ *is known;*

(5) the payoffs $E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} (\boldsymbol{r}_3)^2$ *and* $E_{\boldsymbol{\mu}} (E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3)^2$ *of an ambiguous asset* $j = 3$ *are known and satisfy* $(E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3)^2 \neq E_{\boldsymbol{\mu}} (E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3)^2$*.*

Then, the portfolio indifference correspondence identifies the cardinal index for risk u,and cardinal index for uncertainty ϕ*, up to a positive affine transformation.*

Proof. See the proof in the appendix. □

Remark 19. Also, the coefficient of $\frac{\phi''}{\phi'}$ is the variance of random variable $E_{\boldsymbol{\nu}_a} \boldsymbol{r}_3$ under ambiguity probability measure $\boldsymbol{\mu}$, and the required information on distribution of asset return is the same as in Corollary 2.

Remark 20. Under the same assumptions on asset return and individual belief, both observing individual asset demand and knowing individual portfolio indifference correspondence give the identification result. This should not be surprising, since we can trace out individual asset demand from knowledge of his indifference correspondence.

3.3.2 Identification without any riskless asset

The previous recovery argument requires the existence of one riskless asset and one ambiguity-free asset. Is the identification possible without such assumption? We will relax the assumption gradually, and establish the conditions for identification. In this section, we establish recoverability without any riskless asset, but assume the existence of ambiguity-free assets. Without one riskfree asset, it requires the underlying risk index u being analytic at $x = 0$.

Assumption 4

(1) u is analytic on $\mathbb{R}_+$, and is strictly increasing and concave;

(2) Φ is $\boldsymbol{C}^1$ on $\mathbb{R}_+^A$, is strictly quasi-concave, and satisfies $\forall\ w(x) \in \mathbb{R}_+^A$, $\Phi_a > 0$ for all $a = 1, ..., A$;

(3) $\Phi\left(..., u^{-1}\big(E_{\boldsymbol{\nu}(a)}u(x_s)\big), ...\right)$ is strictly quasi-concave on $\mathbb{R}_+^S$.

Proposition 5. *Suppose that*

(1) the objective function satisfies Assumption 4, and asset return satisfies Assumption 2;

(2) there are two assets $j = 1, 2$ that are ambiguity-free: their payoff distributions are invariant to the states of ambiguity;

(3) the family of conditional probability measures over states of risk, $\boldsymbol{\nu} : \boldsymbol{A} \to \boldsymbol{\Delta(S)}$ is known;

(4) the matrix of expected marginal utility, $\left[..., \frac{E_{\boldsymbol{\nu}(a)}u'(\boldsymbol{Ry})\boldsymbol{R}}{u'\left(u^{-1}\left(E_{\boldsymbol{\nu}(a)}u(\boldsymbol{Ry})\right)\right)}, ...\right]$ *has full row rank A at each portfolio $\boldsymbol{y}$.*

Then, the demand for assets identifies the cardinal index for risk, u, up to a positive affine transformation, as well as the ordinal utility function Φ, up to a monotonically increasing transformation.

Proof. Step 1—recovering risk index:

Restrict attention to portfolios $\boldsymbol{y} = (y_1, y_2, 0, ..., 0)$, and let $\tilde{\boldsymbol{y}} = (y_1, y_2)$ be the associated truncated portfolio.

Since the distribution of payoffs of assets 1 and 2 are invariant across states of ambiguity, there exists a probability measure, $\tilde{\boldsymbol{\nu}} \in \boldsymbol{\Delta(S)}$, and a matrix of payoffs of assets over states of risk $\tilde{\boldsymbol{R}} = (\tilde{\boldsymbol{r}}_1, \tilde{\boldsymbol{r}}_2)$, such that, the distribution of payoffs of assets generated by $(\boldsymbol{\nu}_a, \boldsymbol{Ry})$, for any state of ambiguity, coincides with the distribution generated by $(\tilde{\boldsymbol{\nu}}, \tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})$.

As a consequence,

$$m_{12}(\tilde{\boldsymbol{y}}) = \frac{E_{\tilde{\boldsymbol{\nu}}} u'(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{y}})}{E_{\tilde{\boldsymbol{\nu}}} u'(\tilde{\boldsymbol{R}}(\tilde{\boldsymbol{y}}) \tilde{\boldsymbol{r}}_2} > 0. \tag{3.27}$$

Identification of the cardinal risk index, u, then follows as in Lemma 2.

Step 2—recovering uncertainty index:

Once the risk aversion index u is recovered, the recovery of Φ follows the same argument in Proposition 4. □

Remark 21. Without one riskless asset, the marginal rate of substitution between two ambiguity-free assets identifies the risk aversion index u, however, it requires u being analytic.

Remark 22. The argument in Proposition 5 can be used to restore the recovery result in Corollary 2 and Corollary 3 with no riskless asset. We omit the details.

3.3.3 Identification without any ambiguity-free asset

In this section, we first assume the existence of one riskfree asset, and show that risk and uncertainty preference can be identified without any ambiguity-free asset if we can observe both individual consumption and portfolio choices, and the underlying utility is additively separable. Then we give the identification result when neither one riskfree asset nor one ambiguity-free asset is available.

When individual preference is is additively separable across time, at price of consumption p_0 and prices of assets $\boldsymbol{p}$, the individual optimization problem for joint choice of consumption and portfolio is

$$\max_{c_0 \in \mathbb{R}_+, \boldsymbol{y} \in \mathbb{R}^J} U = u(c_0) + \beta u\Big(\phi^{-1}\big(E_{\boldsymbol{\mu}} \phi(u^{-1}(E_{\boldsymbol{\nu}_a} u(\boldsymbol{Ry})))\big)\Big)$$

$$s.t.\ p_0 c_0 + \boldsymbol{py} = 1.$$

Assumption 5

(1) u is $\boldsymbol{C}^2$ on $\mathbb{R}_{++}$, is strictly concave and satisfies $\forall x \in \mathbb{R}_{++}$, $u^{'} > 0$;

(2) ϕ is $\boldsymbol{C}^2$ on $\mathbb{R}_{++}$, is strictly concave, and satisfies $\forall x \in \mathbb{R}_{++}$, $\phi^{'} > 0$;

(3) $u\Bigg(\phi^{-1}\big(E_{\boldsymbol{\mu}}\phi(u^{-1}(.))\big)\Bigg)$ is strictly concave on $\mathbb{R}_{++}$.

Solutions to the optimization problem, $c_0(p_0, \boldsymbol{p})$ and $\boldsymbol{y}(p_0, \boldsymbol{p})$, exist and are unique. The demand for consumption and assets satisfies necessary and sufficient first order conditions, which will identify $m_{0j}(c_0, \boldsymbol{y})$, the marginal rate of substitution between consumption c_0 and asset j, and $m_{jk}(c_0, \boldsymbol{y})$, the marginal rate of substitution between asset j and asset k.

Proposition 6. *Suppose that*

(1) the objective function is additively separable, and satisfies Assumption 5, and asset return satisfies Assumption 2;

(2) there is an asset that is risk-free: $\boldsymbol{r}_1 = 1$ across states of the world;

(3) the payoffs $E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}(\boldsymbol{r}_2)^2$ and $E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_2)^2$ of an ambiguous asset $j = 2$ are known and satisfy $(E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_2)^2 \neq E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_2)^2$.

Then, the demand for consumption and assets identities the cardinal index for risk u, and the cardinal index for uncertainty ϕ, up to a positive affine transformation.

Proof. Step 1—recovering discount factor:

At $c = c_0$ and $\boldsymbol{y} = (c_0, 0, ..., 0)$, we have

$$\frac{u^{'}(c_0)}{\beta u'(c_0)} = m_{01}(c_0, c_0, 0, ..., 0). \tag{3.28}$$

Thus discount factor is recovered as

$$\beta = \frac{1}{m_{01}(c_0, c_0, 0, ..., 0)}. \tag{3.29}$$

Step 2—recovering risk index:

At $c = c_0$ and $\boldsymbol{y} = (x, 0, ..., 0)$, we have

$$\frac{u^{'}(c_0)}{\beta m_{01}(c_0, x, 0, ..., 0)} = u^{'}(x). \tag{3.30}$$

Integrate both sides with respect to x, we will recover risk index u up to positive affine transformation.

Step 3−recovering uncertainty index:

Marginal rate of substitution between assets 1 and 2 gives

$$E_{\boldsymbol{\mu}}\phi^{'}\left(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\big)\right)\frac{E_{\boldsymbol{\nu}_a}u^{'}(\boldsymbol{Ry})\boldsymbol{r}_1}{u^{'}\left(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\big)\right)}$$
$$= m_{12}(c_0, \boldsymbol{y}) \cdot E_{\boldsymbol{\mu}}\phi^{'}\left(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\big)\right)\frac{E_{\boldsymbol{\nu}_a}u^{'}(\boldsymbol{Ry})\boldsymbol{r}_2}{u^{'}\left(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry})\big)\right)}. \tag{3.31}$$

Take derivative with respect to y_2, and evaluate at $c = c_0$ and $\boldsymbol{y} = (x, 0, ..., 0)$, we have

$$[(E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_2)^2 - E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_2)^2]\frac{\phi^{''}(x)}{\phi^{'}(x)} =$$
$$[E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}(\boldsymbol{r}_2)^2 - E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_2)^2]\frac{u^{''}(x)}{u^{'}(x)} + (E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_2)^2\frac{\partial m_{12}(x, 0, ..., 0)}{\partial y_2}. \tag{3.32}$$

The assumption that $(E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}r_2)^2 \neq E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}r_2)^2$ guarantees that the coefficient of $\frac{\phi^{''}(x)}{\phi^{'}(x)}$ does not vanish, so ϕ can be recovered up to a positive affine transformation. Notice that the recovered ϕ is invariant to any positive affine transformation of u.

□

Remark 23. In the above argument, the requirement on the asset payoffs is relaxed, instead of ambiguity free,the distribution of asset payoffs can be ambiguity state dependent; however, we strengthen the underlying utility functions to restore the recovery result.

Remark 24. In terms of knowledge of the distribution of asset return, the requirement is not more stringent−knowing three moments evaluated by $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, i.e. $E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_2$, $E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}(\boldsymbol{r}_2)^2$, and $E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_2)^2$ suffices.

We can equivalently establish recoverability of risk and uncertainty index from knowledge of individual consumption-portfolio indifference correspondence

$$I(c_0,\boldsymbol{y}) = \bigg\{(c,\boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^J : u(c) + \beta u\bigg(\phi^{-1}\big(E_{\boldsymbol{\mu}}\phi(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Rx}))\big)$$
$$= u(c_0) + \beta u\bigg(\phi^{-1}\big(E_{\boldsymbol{\mu}}\phi(u^{-1}(E_{\boldsymbol{\nu}_a}u(\boldsymbol{Ry}))\big)\bigg)\bigg\}.$$

However, we omit the details.

The last question is: can the identification result be obtained without any riskless and ambiguity-free asset? When all assets are subject to ambiguity, the next result shows that risk index u and ambiguity index ϕ are recoverable if the underlying objective function is additively separable and analytic, we can observe both consumption and portfolio choice, and observe the whole distribution of the asset returns.

Assumption 6

(1) u is analytic on $\mathbb{R}_+$, is strictly concave and satisfies $\forall x \in \mathbb{R}_+$, $u^{'} > 0$;

(2) ϕ is analytic on $\mathbb{R}_+$, is strictly concave, and satisfies $\forall x \in \mathbb{R}_+$, $\phi^{'} > 0$;

(3) $u(\phi^{-1}(E_{\boldsymbol{\mu}}\phi(u^{-1}(.))))$ is strictly concave on $\mathbb{R}_+$.

Corollary 5. *Suppose that*

(1) the objective function is additively separable and satisfy Assumption 6, and asset return satisfies Assumption 2;

(2) the probability measure over states of ambiguity $\boldsymbol{\mu}$ is known;

(3) the family of conditional probability measures over states of risk, $\boldsymbol{\nu}$: $\boldsymbol{A} \to \boldsymbol{\Delta(S)}$ is known.

Then, the demand for consumption and assets identities the cardinal index for risk u, and the cardinal index for uncertainty ϕ,up to a positive affine transformation.

Remark 25. Without riskless asset, the assumption on underlying utility function is much stronger; besides, instead of certain moments of asset return, we need to know the whole distribution for identification.

Remark 26. The above recovery argument requires only one asset, this does not mean the relative asset prices provides no more information (actually they do); however, for recovery one risky asset suffices.

Appendix A.

Proof of Corollary 4:

Proof. Step 1—recovering risk index:

Consider, in portfolio space $\mathbb{R}^J$, the plane $\boldsymbol{\Lambda}_1 = \{\boldsymbol{y} \in \mathbb{R}^J : y_j = 0, j = 3, ..., J\}$. For any point $\overline{\boldsymbol{y}} = (\overline{y_1}, \overline{y_2}, 0, ..., 0)$ in the plane $\boldsymbol{\Lambda}_1$, from the implicit function theorem, in some neighborhood $\aleph_1$ of $(\overline{y_1}, \overline{y_2}, 0, ..., 0)$, y_1 can be written as a unique twice continuously differentiable function $y_1 = f(y_2)$ such that

$$E_{\boldsymbol{\mu}}\phi\bigg(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(f(y_2)\boldsymbol{r}_1 + y_2\boldsymbol{r}_2)\big)\bigg) = \overline{\phi}, \tag{A1}$$

everywhere on $\aleph_1$. This is the parametric expression of individual indifference curve passing through $\overline{\boldsymbol{y}}$ in the plane $\boldsymbol{\Lambda}_1$, and therefore is observable.

Totally differentiate the above equation (A1) with respect to y_2, we have

$$E_{\boldsymbol{\mu}}\phi^{'}\left(u^{-1}\big(Eu_{\boldsymbol{\nu}_a}(f(y_2)\boldsymbol{r}_1 + y_2\boldsymbol{r}_2)\big)\right)\frac{E_{\boldsymbol{\nu}_a}u^{'}(f(y_2)\boldsymbol{r}_1 + y_2\boldsymbol{r}_2)(f^{'}(y_2)\boldsymbol{r}_1 + \boldsymbol{r}_2)}{u^{'}\left(u^{-1}\big(E_{\boldsymbol{\nu}_a}u(f(y_2)\boldsymbol{r}_1 + y_2\boldsymbol{r}_2)\big)\right)} = 0. \tag{A2}$$

Since the payoffs of asset 1 and 2 are invariant to ambiguity states, the above equation (A2) gives

$$E_{\boldsymbol{\nu}_a}u^{'}(f(y_2)\boldsymbol{r}_1 + y_2\boldsymbol{r}_2)(f^{'}(y_2)\boldsymbol{r}_1 + \boldsymbol{r}_2) = 0. \tag{A3}$$

We get

$$f^{'}(\overline{y_2}) = -\frac{E_{\boldsymbol{\nu}_a}u^{'}(f(\overline{y_2})\boldsymbol{r}_1 + \overline{y_2}\boldsymbol{r}_2)\boldsymbol{r}_2}{E_{\boldsymbol{\nu}_a}u^{'}(f(\overline{y_2})\boldsymbol{r}_1 + \overline{y_2}\boldsymbol{r}_2)\boldsymbol{r}_1}. \tag{A4}$$

Further totally differentiate equation (A3) with respect to y_2 , we have

$$E_{\boldsymbol{\nu}_a}u^{''}\big(f(y_2)\boldsymbol{r}_1 + y_2\boldsymbol{r}_2\big)(f^{'}(y_2)\boldsymbol{r}_1 + \boldsymbol{r}_2)^2 + E_{\boldsymbol{\nu}_a}u^{'}\big(f(y_2)\boldsymbol{r}_1 + y_2\boldsymbol{r}_2\big)f^{''}(y_2)\boldsymbol{r}_1 = 0. \tag{A5}$$

We have

$$f^{''}(\overline{y_2}) = -\frac{E_{\boldsymbol{\nu}_a} u^{''}(f(\overline{y_2})\boldsymbol{r}_1 + \overline{y_2}\boldsymbol{r}_2)(f^{'}(\overline{y_2})\boldsymbol{r}_1 + \boldsymbol{r}_2)^2}{E_{\boldsymbol{\nu}_a} u^{'}(f(\overline{y_2})\boldsymbol{r}_1 + \overline{y_2}\boldsymbol{r}_2)\boldsymbol{r}_1}. \tag{A6}$$

At $(\overline{y_1}, \overline{y_2}, 0, ..., 0)$ with $\overline{y_2} = 0$,

$$-\frac{u^{''}(\overline{y_1})}{u^{'}(\overline{y_1})} = \frac{f^{''}(0)}{E_{\boldsymbol{\nu}_a}(f^{'}(0) + \boldsymbol{r}_2)^2}. \tag{A7}$$

Since individual indifference correspondence is observable, so are $f^{'}()$ and $f^{''}()$. Integration like in the above Proposition 4 will identify risk index u uniquely up to a positive affine transformation.

Step 2—recovering uncertainty index:

Consider, in portfolio space $\mathbb{R}^J$, another plane $\boldsymbol{\Lambda}_2 = \{\boldsymbol{y} \in \mathbb{R}^J : y_j = 0, j = 2, 4, ..., J\}$. For any point $\overline{\boldsymbol{y}} = (\overline{y_1}, 0, \overline{y_3}, 0, ..., 0)$ in the plane $\boldsymbol{\Lambda}_2$, by the implicit function theorem, in some neighborhood $\aleph_2$ of $(\overline{y_1}, 0, \overline{y_3}, 0, ..., 0)$, y_1 can be written as a unique twice continuously differentiable function $y_1 = f(y_3)$ such that:

$$E_{\boldsymbol{\mu}}\phi\Big(u^{-1}\big(E_{\boldsymbol{\nu}_a} u(f(y_3)\boldsymbol{r}_1 + y_3\boldsymbol{r}_3)\big)\Big) = \overline{\phi}, \tag{A8}$$

everywhere on $\aleph_2$. As the parametric form of individual indifference curve passing through $\overline{y}$ in the plane $\boldsymbol{\Lambda}_2$, it is observable.

Totally differentiate the above equation (A8) with respect to y_3, we have

$$E_{\boldsymbol{\mu}}\phi^{'}\left(u^{-1}\big(Eu_{\boldsymbol{\nu}_a}(f(y_3)\boldsymbol{r}_1 + y_3\boldsymbol{r}_3)\big)\right)\frac{E_{\boldsymbol{\nu}_a} u^{'}\big(f(y_3)\boldsymbol{r}_1 + y_3\boldsymbol{r}_3\big)(f^{'}(y_3)\boldsymbol{r}_1 + \boldsymbol{r}_3)}{u^{'}\left(u^{-1}\big(E_{\boldsymbol{\nu}_a} u(f(y_3)\boldsymbol{r}_1 + y_3\boldsymbol{r}_3)\big)\right)} = 0. \tag{A9}$$

At $(\overline{y_1}, 0, \overline{y_3}, 0, ..., 0)$ with $\overline{y_3} = 0$, we have $f^{'}(0) = -E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3$.

Further differentiate equation (A9) with respect to y_3, and evaluate at

$(\overline{y_1}, 0, \overline{y_3}, 0, ..., 0)$ with $\overline{y_3} = 0$, we get

$$[E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3 - E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3)^2]\frac{\phi^{''}(\overline{y_1})}{\phi^{'}(\overline{y_1})} =$$

$$[E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3 - E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3)^2 - E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}(\boldsymbol{r}_3 - E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}_a}\boldsymbol{r}_3)^2]\frac{u^{''}(\overline{y_1})}{u^{'}(\overline{y_1})} + f^{''}(0). \quad \text{(A10)}$$

Given risk index u recovered, the right hand side is observable, and is invariant to any positive affine transformation of risk index u. Integration will identify uncertainty index ϕ uniquely up to a positive affine transformation. □

Chapter 4

Bounding risk and ambiguity aversion

4.1 Introduction

The standard framework for understanding consumer choice under uncertainty is expected utility theory due to von Neumann and Morgenstern (1944), Savage (1954), and Anscombe and Aumann (1963). The implication of expected utility theory for individual choice and market equilibrium, and its comparative statics crucially rely on the shape of expected utility index, see Gollier (2001) for detailed discussion. Varian (1988), in particular, uses the revealed preference approach introduced by Afriat (1967) and Diewert (1973) to derive the necessary and sufficient conditions consistent with different shapes of consumer's risk preference, i.e. whether the absolute (relative) risk aversion is increasing or decreasing with wealth. If the observed data passes these conditions, they can give both absolute and relative risk aversion a lower bound and an upper bound. Such a nonparametric approach does not specify any functional form about the consumer's utility function.

However, experimental evidence shows that consumers may not maximize expected utility when they do not know the objective probability distribution. The decision theory literature has developed different utility models under ambiguity to accommodate such behavior observed in experiments. The smooth ambiguity model due to Klibanoff, Marinacci, and Mukerji (2005) has drawn more and more attention. Collard, Mukerji, Sheppard, and Tallon (2011), Ju and Miao (2012) show that the smooth ambiguity model has the potential to explain the equity premium puzzle. Guerdjikova and Sciubba (2015) shows that the ambiguity aversion matters for survival in the financial market. These papers also show that the implication of the smooth ambiguity model crucially depends on the shape of the risk and ambiguity aversion and their magnitudes. However, currently there is very little work on testing the shape of the risk and ambiguity aversion and estimating their magnitudes.

In this paper, I will revisit Varian (1988), and rewrite Varian's conditions in terms of Afriat numbers, which can be used to test the portfolio choice from incomplete markets. Then I will extend Varian's argument to ambiguity case: to use the nonparametric method to derive the necessary and sufficient conditions to be compatible with different shapes of risk and ambiguity aversion; if the data passes these conditions, bounds on the risk and ambiguity aversion can be derived.

4.2 Bounding risk aversion under pure risk

First, I consider the pure risk case, where the probability distribution over these states is objectively known, represented by a vector $\boldsymbol{\pi} \in \mathbb{R}^S_{++}$. Assume the consumer has expected utility, and his utility index $u(x)$ is strictly increasing, strictly concave, and twice differentiable.

4.2.1 Varian's bound

Varian (1988) assumes complete contingent consumption x_s for each state, and the consumer's maximization problem is

$$\max_{\boldsymbol{x} \in \mathbb{R}^S_{++}} U(\boldsymbol{x}) = \sum_{s=1}^{S} \pi_s u(x_s) \;\; s.t. \; \mathbf{p} \cdot \mathbf{x} \leq I. \tag{4.1}$$

Varian (1988) assumes the analyst has access to one observation $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\pi})$ only, and assume the consumptions in each state are different, i.e. $x_s \neq x_{s'}$ for $s \neq s'$. When will such choice data be consistent with strictly concave expected utility maximization? Proposition 7 gives the necessary and sufficient conditions.

Proposition 7. *The following conditions are equivalent:*

(i) *The single observation* $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\pi})$ *is generated from strictly concave expected utility maximization.*

(ii) *There exist numbers* U_s*, s=1...S, satisfying the following Afriat inequalities:*

$$U_s < U_{s'} + \frac{p_{s'}}{\pi_{s'}}(x_s - x_{s'}), \text{ where } s \neq s'.$$

(iii) *Any two pairs of* $\{(x_s, \frac{p_s}{\pi_s}) : s \in S\}$ *satisfy,*

$$(\frac{p_s}{\pi_s} - \frac{p_{s'}}{\pi_{s'}})(x_{s'} - x_s) > 0, \ \forall \ s,s' \in \{1,\ldots,S\}, \text{ and } s \neq s'.$$

Green and Srivastava (1986) derive similar conditions for concave expected utility maximization for multiple observations where there are multiple goods in each state. Chiappori and Rochet (1987), and Matzkin and Richter (1991) give necessary and sufficient conditions for strictly concave rationality for multiple observations. Neither of their arguments works for strictly concave expected utility rationalizing one observation. Kubler and Schmedders (2010) give a proof for rationalizing one observation under complete markets. In Proposition 8 below, I modify their proof to construct strictly concave expected utility to rationalize one observation in incomplete markets, then this proposition will follow directly.

Proof. The proof follows directly from the argument in Proposition 8. □

For a general utility function, at least two observations are needed to refute utility maximization hypothesis; however, under expected utility, the stationarity of risk aversion index $u(x)$ generates further restrictions on the data. And just one observation can refute the expected utility hypothesis.

Example 3. Suppose there are 2 states with equal probabilities, i.e. $\pi_1 = \pi_2 = \frac{1}{2}$. Suppose consumer's portfolio choice under price $\boldsymbol{p} = (1,1)$ is $\boldsymbol{x} = (1,2)$. It can be verified that there is no solution for inequalities in condition (ii), nor does the data satisfy condition (iii), so such choice can not be rationalized by a strictly concave expected utility; however, it is consistent with a concave expected utility.

Note that the testable restriction in Example 1 comes from stationarity of index u rather than its concavity. As in the case of any general utility function, concavity does not have testable implication under linear budget constraint.[1] To see this, suppose the utility index is state dependent, then consumer's expected utility can be written as $\sum_{s=1}^{S} u_s(x_s)$, where $u_s(x_s)$ is concave for each s. In this case, one observation can never refute the hypothesis of maximizing $\sum_{s=1}^{S} u_s(x_s)$.

Remark 27. Under one observation from complete markets, condition (iii) in Proposition 7 is quantifier free, and can be be checked against the data $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\pi})$ directly. This is not true under multiple observations or incomplete markets.

In Varian (1988), the question asked is: what further conditions must the single observation $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\pi})$ be satisfied if it's consistent with decreasing (increasing) absolute (relative) risk aversion? Now let's number the states of nature in the way such that $x_1 < x_2 < ... < x_S$. Denote by $r^A(x) = -\frac{u^{''}(x)}{u^{'}(x)}$ the Arrow-Pratt measure of absolute risk aversion, and $r^R(x) = -x\frac{u^{''}(x)}{u^{'}(x)}$ the relative risk aversion. When the average risk aversion in an economy is considered, it is assumed that there are n consumers and denote by X_s the aggregate consumption in state s, i.e. $X_s = \sum_{i=1}^{n} x_s^i$. Denote by $R^A(X_s) = (\sum_{i=1}^{n} \frac{1}{r_i^A(x_s^i)})^{-1}$ the average absolute risk aversion, and $R^R(X_s) = (\sum_{i=1}^{n} \frac{1}{r_i^R(x_s^i)})^{-1}$ the average relative risk aversion.

As pointed out by Varian (1988), the decreasing (increasing) absolute risk aversion is equivalent to the requirement that $\log u^{'}(x)$ is a convex (concave) function of x, and the increasing (decreasing) relative risk aversion is equivalent to $\log u^{'}(x)$ being a concave (convex) function of $\log x$. Varian (1988) gives the necessary and sufficient conditions for expected utility displaying such properties, and shows that such properties can be aggregated across individuals under complete Arrow security markets.

Theorem 1 (Varian, 1988)**.** *Suppose the single observation* $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\pi})$ *satisfies the conditions in the Proposition 7. Then the following results hold:*

(i) *Decreasing absolute risk aversion is equivalent to the following Varian*

[1] Concavity does have testable implication under nonlinear budget constraint, see Cherchye, Demuynck, and Rock (2014).

Ratio Condition:

$$\frac{\log \frac{p_s}{\pi_s} - \log \frac{p_{s+1}}{\pi_{s+1}}}{x_{s+1} - x_s} \leq r^A(x_s) \leq \frac{\log \frac{p_{s-1}}{\pi_{s-1}} - \log \frac{p_s}{\pi_s}}{x_s - x_{s-1}}.$$

(ii) *Decreasing relative risk aversion is equivalent to the following Varian Relative Ratio Condition:*

$$\frac{\log \frac{p_s}{\pi_s} - \log \frac{p_{s+1}}{\pi_{s+1}}}{\log x_{s+1} - \log x_s} \leq r^R(x_s) \leq \frac{\log \frac{p_{s-1}}{\pi_{s-1}} - \log \frac{p_s}{\pi_s}}{\log x_s - \log x_{s-1}}.$$

(iii) *Suppose each consumer has decreasing absolute risk aversion, then decreasing average absolute risk aversion is equivalent to the following Varian Aggregate Ratio Condition:*

$$\frac{\log \frac{p_s}{\pi_s} - \log \frac{p_{s+1}}{\pi_{s+1}}}{X_{s+1} - X_s} \leq R^A(X_s) \leq \frac{\log \frac{p_{s-1}}{\pi_{s-1}} - \log \frac{p_s}{\pi_s}}{X_s - X_{s-1}}.$$

(iv) *Suppose each consumer has decreasing relative risk aversion, then decreasing average absolute risk aversion is equivalent to Varian Aggregate Relative Ratio Condition:*

$$\frac{\log \frac{p_s}{\pi_s} - \log \frac{p_{s+1}}{\pi_{s+1}}}{\log X_{s+1} - \log X_s} \leq R^R(X_s) \leq \frac{\log \frac{p_{s-1}}{\pi_{s-1}} - \log \frac{p_s}{\pi_s}}{\log X_s - \log X_{s-1}}.$$

Remark 28. In the above proposition, all the conditions are stated for *decreasing* absolute (relative) risk aversion; however, if the inequalities are reversed, they will become necessary and sufficient conditions for *increasing* absolute (relative) risk aversions.

Remark 29. Conditions (iii) and (iv) show that consumers' preferences can be aggregated. This is not surprising, since it is well known that under complete market there exists a representative consumer (Dybvig and Ross (2003)).

One possible extension is to consider incomplete markets, i.e. fewer assets than states of the nature. Assume there are J assets, and asset j has payoff r_{sj} in state s which is nonnegative. I assume that the payoff vectors of these J assets are linearly independent, i.e. there is no redundant asset. Denote by P_j the price of asset j, and y_j the amount of asset j bought by the consumer. Then the consumption in state s will be $x_s = \sum_{j=1}^{S} r_{sj} y_j$.

The non-arbitrage of asset prices implies the existence of "implicit Arrow−Debreu prices" $(p_s)_{s=1}^{S}$ such that

$$\sum_{s=1}^{S} r_{sj} p_s = P_j. \tag{4.2}$$

Since there are less assets than the states of nature, the payoff matrix has rank J, so the above equations (4.2) have multiple solutions. If any such implied prices $(p_s)_{s=1}^{S}$ and $(x_s)_{s=1}^{S}$ satisfy conditions in Proposition 7, then they (after being normalized by probabilities) can serve as marginal utilities in consumer's first-order conditions. And the data $(\boldsymbol{p}, \boldsymbol{x})$ can be used to bound consumer's risk aversion as in Theorem 1.

Corollary 6. *The single portfolio choice* $(\boldsymbol{P}, \boldsymbol{y}, \boldsymbol{\pi})$ *is compatible with expected utility displaying decreasing absolute (relative) risk aversion, if and only if there exist prices* $(p_s)_{s=1}^{S}$ *satisfying equations (4.2) and the pair* $(p_s, x_s)_{s=1}^{S}$ *satisfies conditions in Proposition 7 and the ratio conditions in (i) and (ii) of Theorem 1.*

Remark 30. The aggregation result cannot be obtained under incomplete market, since the equations (4.2) have multiple solutions, and the implied Arrow−Debreu prices (or marginal utilities) will be different across consumers. Under complete markets, such prices are unique.

4.2.2 Afriat's bound

Varian's bounds on absolute (relative) risk aversion are based on the "cyclic monotonicity" properties of one single observation data, which are equivalent to Afriat's inequalities under complete markets. The equivalence of Afriat inequalities and "cyclic monotonicity" still holds for multiple observations under incomplete markets. I assume that the consumption levels differ across states s and dates t, i.e. $\sum_{j=1}^{J} r_{sj} y_j^t \neq \sum_{j=1}^{J} r_{s'j} y_j^{t'}$ for either $s \neq s'$ or $t \neq t'$.

Proposition 8. *The following conditions are equivalent:*

(i) *The observations* $(\boldsymbol{P}^t, \boldsymbol{y}^t, \boldsymbol{\pi}^t)^{t=1\ldots T}$ *are generated from strictly concave expected utility maximization.*

(ii) *There exist numbers* $(U_s^t)_{s=1\ldots S}^{t=1\ldots T}$*,* $(M_s^t)_{s=1\ldots S}^{t=1\ldots T} > 0$*, and* $(\lambda^t)^{t=1\ldots T} > 0$*, satisfying the following conditions:*

$$U_s^t < U_{s'}^{t'} + M_{s'}^{t'}(\sum_{j=1}^{J} r_{sj} y_j^t - \sum_{j=1}^{J} r_{s'j} y_j^{t'}),$$

$$\sum_{s=1}^{S} \pi_s^t M_s^t r_{sj} = \lambda^t P_j^t.$$

(iii) *There exist numbers* $(M_s^t)_{s=1\ldots S}^{t=1\ldots T} > 0$*, and* $(\lambda^t)^{t=1\ldots T} > 0$*, such that:*

$$(M_s^t - M_{s'}^{t'})(\sum_{j=1}^{J} r_{s'j} y_j^{t'} - \sum_{j=1}^{J} r_{sj} y_j^t) > 0,$$

$$\sum_{s=1}^{S} \pi_s^t M_s^t r_{sj} = \lambda^t P_j^t.$$

Proof. I prove this proposition by showing that (i) implies (ii), (ii) implies (iii), and (iii) implies (i).

(i) implies (ii):

Since utility function is strictly concave, and the consumption levels are different across states and dates, the following conditions hold:

$$u(\sum_{j=1}^{J} r_{sj} y_j^t) < u(\sum_{j=1}^{J} r_{s'j} y_j^{t'}) + u'(\sum_{j=1}^{J} r_{s'j} y_j^{t'})(\sum_{j=1}^{J} r_{sj} y_j^t - \sum_{j=1}^{J} r_{s'j} y_j^{t'}). \quad (4.3)$$

The first order condition for maximization is:

$$\sum_{s=1}^{S} \pi_s^t u'(\sum_{j=1}^{J} r_{sj} y_j^t) r_{sj} = \lambda^t P_j^t. \quad (4.4)$$

Put $U_s^t = u(\sum_{j=1}^{J} r_{sj} y_j^t)$, $M_s^t = u'(\sum_{j=1}^{J} r_{sj} y_j^t)$, and substitute into above equations (4.3) and (4.4) to get the conditions in (ii).

(ii) implies (iii):

For any pair of $\{U_s^t, M_s^t : s = 1, ..., S, t = 1, ..., T\}$,

$$U_s^t < U_{s'}^{t'} + M_{s'}^{t'}(\sum_{j=1}^{J} r_{sj} y_j^t - \sum_{j=1}^{J} r_{s'j} y_j^{t'}), \tag{4.5}$$

$$U_{s'}^{t'} < U_s^t + M_s^t(\sum_{j=1}^{J} r_{s'j} y_j^{t'} - \sum_{j=1}^{J} r_{sj} y_j^t). \tag{4.6}$$

Adding the above two inequalities gives the conditions in (iii).

(iii) implies (i):

Denote by $x_s^t = \sum_{j=1}^{J} r_{sj} y_j^t$ the consumption in state s at observation t. I number the observations across states and observations such that $x_1 < ... < x_i < ... < x_{TS}$, where $x_i = x_s^t$ for some $s \in \{1, ...S\}$, and $t \in \{1, ..., T\}$. Then condition (iii) in Proposition 8 implies that $M_1 > ... > M_i > ... > M_{TS}$, where $M_i = M_s^t$ for some $s \in \{1, ...S\}$, and $t \in \{1, ..., T\}$. It can be shown that M_i is the super-gradient of some strictly increasing and concave utility at the observation x_i. I will construct a strictly increasing and strictly concave function $u(x)$ on the whole domain.

On the interval $[x_i, x_{i+1}]$, define

$$u(x) = \frac{(x - x_i)^2}{2(x_{i+1} - x_i)} M_{i+1} - (\frac{x_{i+1} - x_i}{2}) \left(1 - \frac{x - x_i}{x_{i+1} - x_i}\right)^2 M_i + c_i, \tag{4.7}$$

where c_i is some constant.

Since the numbers M_i, $i \in \{1, ..., TS\}$, are strictly positive and strictly decreasing, the constructed function $u(x)$ has the following properties:

$u(x)$ is strictly increasing, since $u^{'}(x) = \frac{x - x_i}{x_{i+1} - x_i} M_{i+1} + (1 - \frac{x - x_i}{x_{i+1} - x_i}) M_i > 0$;

$u(x)$ is strictly concave, since $u^{''}(x) = \frac{1}{x_{i+1} - x_i}(M_{i+1} - M_i) < 0$;

M_i is the super-gradient of $u(x)$ at x_i, since $u^{'}(x_i) = M_i$.

The value of constants c_i can be chosen such that successive pieces of the graph of the constructed function are connected. Then the whole function would be continuous, strictly increasing and strictly concave with derivative M_i at x_i. It can be shown that this function will rationalize the observation. ∎

Unlike the conditions in Proposition 7, the necessary and sufficient conditions in Proposition 8 are not quantifier free, i.e. they require the existence of suitable unknown numbers. But the conditions in Proposition 8 are linear in the unknowns, and can be solved by linear programming with efficient algorithms.

Example 4. The restrictions in Proposition 8 are not vacuous with even one observation: Suppose there are 3 states with equal probabilities, i.e. $\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$. There are two asset with payoffs $\boldsymbol{r}_1 = (1, 1, 1)$ and $\boldsymbol{r}_2 = (0, 0, 6)$. Suppose at price $P_1 = 1$ and $P_2 = 3$, the consumer chooses $y_1 = 1$ and $y_2 = 1$. Then it can be verified that no strictly concave expected utility can rationalize such choice.

If the observed portfolio choice is generated by some strictly concave expected utility, the Afriat numbers $\{U_s^t, M_s^t, \lambda^t : s = 1, ..., S, t = 1, ..., T\}$ should satisfy the conditions in Proposition 8. What further conditions should these Afriat numbers satisfy if the observed data is compatible with particular shapes of risk aversion? Proposition 9 gives the necessary and sufficient conditions.

Proposition 9. *Suppose the observations $(\boldsymbol{P}^t, \boldsymbol{y}^t, \boldsymbol{\pi}^t)^{t=1,...,T}$ are generated from strictly concave expected utility maximization, i.e. there exist numbers U_s^t, $M_s^t > 0$ and $\lambda^t > 0$ satisfying the conditions in Proposition 8. Then the following results hold:*

(i) *Decreasing absolute risk aversion is equivalent to the following Afriat Ratio Condition:*

$$\frac{\log M_{i+1} - \log M_i}{x_i - x_{i+1}} \le r^A(x_i) \le \frac{\log M_i - \log M_{i-1}}{x_{i-1} - x_i}.$$

(ii) *Decreasing relative risk aversion is equivalent to the following Afriat Relative Ratio Condition:*

$$\frac{\log M_{i+1} - \log M_i}{\log(x_i) - \log(x_{i+1})} \le r^R(x_i) \le \frac{\log M_i - \log M_{i-1}}{\log(x_{i-1}) - \log(x_i)}.$$

Proof. As pointed out by Varian (1988), decreasing (increasing) absolute risk aversion is equivalent to the requirement that $\log u^{'}(x)$ is a convex (concave function) function of x, since

$$\frac{d\log u^{'}(x)}{dx}=\frac{u^{''}(x)}{u^{'}(x)}=-r^{A}(x). \tag{4.8}$$

Also, decreasing (increasing) relative risk aversion is equivalent to the requirement that $\log u^{'}(x)$ is a convex (concave function) function of $\log(x)$, since

$$\frac{d\log u^{'}(x)}{d\log(x)}=x\frac{u^{''}(x)}{u^{'}(x)}=-r^{R}(x). \tag{4.9}$$

Here, I only prove (i), since the proof of (ii) will follow the same line.

Necessity:

Since $\log u^{'}(x)$ is a convex function of x, it must satisfy the following inequalities:

$$\log u^{'}(x_{i+1})\geq \log u^{'}(x_i)+\frac{d\log u^{'}(x_i)}{dx_i}[x_{i+1}-x_i], \tag{4.10}$$

$$\log u^{'}(x_{i-1})\geq \log u^{'}(x_i)+\frac{d\log u^{'}(x_i)}{dx_i}[x_{i-1}-x_i]. \tag{4.11}$$

From inequality (4.10),

$$\frac{d\log u^{'}(x_i)}{dx_i}\leq\frac{\log\frac{u^{'}(x_{i+1})}{u^{'}(x_i)}}{x_{i+1}-x_i}. \tag{4.12}$$

From inequality (4.13),

$$\frac{d\log u^{'}(x_i)}{dx_i}\geq\frac{\log\frac{u^{'}(x_i)}{u^{'}(x_{i-1})}}{x_i-x_{i-1}}. \tag{4.13}$$

Therefore

$$\frac{\log\frac{u^{'}(x_{i+1})}{u^{'}(x_i)}}{x_i-x_{i+1}}\leq r^{A}(x_i)\leq\frac{\log\frac{u^{'}(x_i)}{u^{'}(x_{i-1})}}{x_{i-1}-x_i}. \tag{4.14}$$

Since the observed data is rationalized by some strictly concave expected utility function, then from Afriat inequality,

$$u'(x_i) = M_i. \tag{4.15}$$

Then the following must hold

$$\frac{\log M_{i+1} - \log M_i}{x_i - x_{i+1}} \leq r^A(x_i) \leq \frac{\log M_i - \log M_{i-1}}{x_{i-1} - x_i}. \tag{4.16}$$

Sufficiency:

Pick a set of numbers r_i^A satisfying the *Afriat Ratio Condition.*

Define the function

$$u(x) = \int_0^x M(x)dx = \int_0^x exp[\log M(x)]dx, \tag{4.17}$$

where

$$\log M(x) = \max_i\{\log M_i - r_i^A(x - x_i)\}. \tag{4.18}$$

Suppose the Afriat numbers satisfy the *Afriat inequalities* and *Afriat Ratio Condition.* Then I will show the constructed overall utility index $U(\boldsymbol{z})$ will rationalize the data $(\boldsymbol{P}^t, \boldsymbol{y}^t, \boldsymbol{\pi}^t)_{t=1,...,T}$, and exhibit decreasing absolute risk aversion.

From the construction, $\log M(x)$ is differentiable with respect to x at the observation x_i, since it is assumed $x_1 < ... < x_{TS}$. The following holds:

$$\frac{d \log M(x_i)}{dx} = \frac{M'(x_i)}{M(x_i)} = -r_i^A. \tag{4.19}$$

The first derivative of $U(\boldsymbol{z})$ at y_j^t is $\sum_{s=1}^S \pi_s^t M(\sum_{j=1}^J r_{sj} y_j^t) r_{sj}$.
I claim that $M(x_i) = M(\sum_{j=1}^J r_{sj} y_j^t) = M_i$, since

$$\begin{aligned}
\log M(x_i) &= \max_{h \in \{1,...,TS\}} \{\log M_h - r_h(x_i - x_h)\} \\
&= \log M_m - r_m(x_i - x_m) \\
&\geq \log M_i - r_i(x_i - x_i) \\
&= \log M_i
\end{aligned}$$

And the above inequality cannot be strict, otherwise it will violate the

Afriat Ratio Condition.

Since the Afriat numbers satisfy the conditions $\sum_{s=1}^{S} \pi_s^t M_s^t r_{sj} = \lambda^t P_j^t$ in Proposition 8, then the following condition hold:

$$\sum_{s=1}^{S} \pi_s^t M(\sum_{j=1}^{J} r_{sj} y_j^t) r_{sj} = \lambda^t P_j^t. \tag{4.20}$$

The second derivative of $U(\boldsymbol{y})$ at y_j is

$$\sum_{s=1}^{S} \pi_s^t M'(\sum_{j=1}^{J} r_{sj} y_j^t) r_{sj}^2 = -\sum_{s=1}^{S} \pi_s^t r_s^t M(\sum_{j=1}^{J} r_{sj} z_j^t) r_{sj}^2 < 0. \tag{4.21}$$

Thus $U(\boldsymbol{y})$ is a concave function. So the satisfaction of the first-order conditions is a sufficient condition for the observed choice to solve the maximization problem.

Finally, the absolute risk aversion at x_i is given by

$$r^A(x_i) = -\frac{u''(x_i)}{u'(x_i)} = r_i^A, \tag{4.22}$$

which is a decreasing sequence in i by construction.

□

The proof of sufficiency modifies the argument of Varian (1988). Actually a simpler proof can be given using the following construction:

Define the function around x_i,

$$\log M(x) = -r_i^A x + \log M_i + r_i^A x_i. \tag{4.23}$$

It can be checked that such function will rationalize the observation and display decreasing absolute risk aversion.

Remark 31. Afriat's bounds for risk aversion are defined in terms of Afriat numbers; however, Afriat's bounds are equivalent to Varian's bounds. Under complete markets, $M_s = \frac{p_s}{\pi_s}$; under incomplete markets, the right hand side is the implied Arrow−Debreu prices. Under complete markets, the aggregation result would be obvious.

4.3 Estimating risk and ambiguity aversion: Afriat's bounds

Under the smooth ambiguity model, a consumer will solve the following maximization problem:

$$\max_{\boldsymbol{y}\in\mathbb{R}^J} \sum_{a=1}^{A} \mu_a \phi\left(\sum_{s=1}^{S} \nu_{as} u\left(\sum_{j=1}^{J} r_{sj} y_j\right)\right) \; s.t.\; \mathbf{P}\cdot\mathbf{y} \le I \tag{4.24}$$

where $\boldsymbol{\nu}_a$ and $\boldsymbol{\mu}$ are probabilities for risk and ambiguity, respectively.

Lemma 5. *The following conditions are equivalent:*

(i) *The observations* $(\boldsymbol{P}^t, \boldsymbol{y}^t, \boldsymbol{\nu}^t, \boldsymbol{\mu}^t)^{t=1,...,T}$ *are generated from strictly concave smooth ambiguity utility maximization.*

(ii) *There exist real numbers* $(U_s^t, M_s^t)_{s=1,...,S}^{t=1,...,T} > 0$, $(\Phi_a^s)_{a=1,...,A}^{s=1,..,S}$, $(\rho_a^t)_{a=1,...,A}^{t=1,...,T} > 0$ *and* $(\lambda^t)^{t=1,...,T} > 0$ *such that for all* $s, s' \in \{1, 2, ..., S\}$, $a, a' \in \{1, 2, ..., A\}$, $t, t^{'} \in \{1, 2, ..., T\}$ *and* $j \in \{1, 2, ..., J\}$

$$U_s^t - U_{s'}^{t'} < M_{s'}^{t'}\left(\sum_{j=1}^{J} r_{sj} y_j^t - \sum_{j=1}^{J} r_{s'j} y_j^{t'}\right),$$

$$\Phi_a^t - \Phi_{a'}^{t'} < \rho_{a'}^{t'}\left(\sum_{s=1}^{S} \nu_{as}^t U_s^t - \sum_{s=1}^{S} \nu_{a's}^{t'} U_s^{t'}\right),$$

and

$$\sum_{a=1}^{A} \mu_a^t \left(\rho_a^t \sum_{s=1}^{S} \nu_{as}^t M_s^t r_{sj}\right) = \lambda^t P_j^t.$$

(iii) *There exist real numbers* $(U_s^t, M_s^t)_{s=1,...,S}^{t=1,...,T} > 0$, $(\rho_h^t)_{a=1,...,A}^{t=1,...,T} > 0$ *and* $(\lambda^t)^{t=1,...,T} > 0$ *such that for all* $s, s' \in \{1, 2, ..., S\}$, $a, a' \in \{1, 2, ..., A\}$, $t, t^{'} \in \{1, 2, ..., T\}$ *and* $j \in \{1, 2, ..., J\}$

$$(M_s^t - M_{s'}^{t'})(\sum_{j=1}^{J} r_{s'j} y_j^{t'} - \sum_{j=1}^{J} r_{sj} y_j^t) > 0,$$

$$(\rho_a^t - \rho_{a'}^{t'})(\sum_{s=1}^{S} \nu_{a's}^{t'} U_s^{t'} - \sum_{s=1}^{S} \nu_{as}^t U_s^t) > 0,$$

and

$$\sum_{a=1}^{A} \mu_a^t \left(\rho_a^t \sum_{s=1}^{S} \nu_{as}^t M_s^t r_{sj} \right) = \lambda^t P_j^t.$$

Remark 32. These conditions are necessary and sufficient for the observed asset demands to be rationalized by some state independent, strictly concave smooth ambiguity preference. Note that these conditions are applicable to one observation, where the restrictions will be across different risk and ambiguity states.

Example 5. This example shows that the above conditions are not vacuous with even one observation: Suppose there are 3 states, and the consumer has ambiguity over the probability distribution. Suppose there are two equally possible probability distributions: $\boldsymbol{\nu}_1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ and $\boldsymbol{\nu}_2 = (\frac{1}{6}, \frac{2}{6}, \frac{1}{2})$. There are two assets with payoffs $\boldsymbol{r}_1 = (1, 1, 1)$ and $\boldsymbol{r}_2 = (0, 0, 6)$. Suppose at price $P_1 = 1$ and $P_2 = 3$, the consumer chooses $y_1 = 1$ and $y_2 = 1$. Then it can be verified that such choice is not consistent with any strictly concave smooth ambiguity utility.

From now on, I will assume that the observed consumption and price data can be rationalized by some smooth ambiguity preference, i.e. there exist Afriat numbers $(U_s^t, M_s^t)_{s=1,...,S}^{t=1,...,T} > 0$, $(\Phi_h^t)_{a=1,...,A}^{t=1,...,T}$, $(\rho_a^t)_{a=1,...,A}^{t=1,...,T} > 0$ and $(\lambda^t)^{t=1,...,T}$ satisfying the restrictions in Lemma 5.

Denote by $x_s^t = \sum_{j=1}^{J} r_{sj} y_j^t$ the consumption in state s at observation t, $u_a^t = \sum_{s=1}^{S} \nu_{as}^t U_s^t$ the expected utility under probability ν_a at observation t. Number the consumption across risk states and observations such that $x_1 < ... < x_i < ... < x_{TS}$, where $x_i = x_s^t$ for some $s \in \{1, ..., S\}$ and $t \in \{1, ..., T\}$, then condition (iii) in Lemma 5 implies that $M_1 > ... > M_i > ... > M_{TS}$, where $M_i = M_s^t$ for some $s \in \{1, ..., S\}$ and $t \in \{1, ..., T\}$. Similarly, number the expected utility across ambiguity states and observations such that $u_1 < ... < u_i < ... < u_{TA}$, where $u_i = u_a^t$ for some $a \in \{1, ..., A\}$ and $t \in \{1, ..., T\}$, it follows that $\rho_1 > ... > \rho_i > ... > \rho_{TA}$, where $\rho_i = \rho_a^t$ for some $a \in \{1, ..., A\}$ and $t \in \{1, ..., T\}$. Denote by $a^A(u) = -\frac{\phi''(u)}{\phi'(u)}$ the absolute ambiguity aversion, and $a^R(u) = -u\frac{\phi''(u)}{\phi'(u)}$ the relative ambiguity aversion.

As in Varian (1988), the question asked is: what further conditions must be satisfied if the observed data $(\boldsymbol{P}^t, \boldsymbol{y}^t, \boldsymbol{\nu}^t, \boldsymbol{\mu}^t)^{t=1,...,T}$ is compatible with various hypotheses about the behavior of risk and ambiguity aversion?

Proposition 10. *Suppose the observations* $(\boldsymbol{P}^t, \boldsymbol{y}^t, \boldsymbol{\nu}^t, \boldsymbol{\mu}^t)^{t=1,\ldots,T}$ *are generated from a strictly concave smooth ambiguity utility, i.e. there exist real numbers* $(U_s^t, M_s^t)_{s=1,\ldots,S}^{t=1,\ldots,T} > 0$, $(\Phi_a^t)_{a=1,\ldots,A}^{t=1,\ldots,T}$, $(\rho_a^t)_{a=1,\ldots,A}^{t=1,\ldots,T} > 0$ *and* $(\lambda^t)^{t=1,\ldots,T} > 0$ *satisfying the conditions in Lemma 5. Then the following results hold:*

(i) *Decreasing absolute risk aversion is equivalent to the following Afriat Ratio Condition i:*

$$\frac{\log M_{i+1} - \log M_i}{x_i - x_{i+1}} \leq r^A(x_i) \leq \frac{\log M_i - \log M_{i-1}}{x_{i-1} - x_i}.$$

(ii) *Decreasing absolute ambiguity aversion is equivalent to the following Afriat Ratio Condition ii:*

$$\frac{\log \rho_{i+1} - \log \rho_i}{u_i - u_{i+1}} \leq a^A(u_i) \leq \frac{\log \rho_{i-1} - \log \rho_i}{u_i - u_{i-1}}.$$

(iii) *Decreasing relative risk aversion is equivalent to the following Afriat Relative Ratio Condition i:*

$$\frac{\log M_{i+1} - \log M_i}{\log(x_i) - \log(x_{i+1})} \leq r^R(x_i) \leq \frac{\log M_i - \log M_{i-1}}{\log(x_{i-1}) - \log(x_i)}.$$

(iv) *Decreasing relative ambiguity aversion is equivalent to the following Afriat Relative Ratio Condition ii:*

$$\frac{\log \rho_{i+1} - \log \rho_i}{\log(u_i) - \log(u_{i+1})} \leq a^R(u_i) \leq \frac{\log \rho_{i-1} - \log \rho_i}{\log(u_i) - \log(u_{i-1})}.$$

Proof. Necessity:

The conditions for decreasing absolute (relative) risk aversion follow from the argument in Proposition 9. And such argument applies to ambiguity aversion: decreasing (increasing) relative ambiguity aversion is equivalent to $\log \phi^{'}(u)$ being a convex (concave) function of $\log u$. In the following, I only show the condition for decreasing relative ambiguity aversion.

Since $\log \phi^{'}(u)$ is a convex function with respect to $\log u$, the following inequalities hold:

$$\log \phi^{'}(u_{i+1}) \geq \log \phi^{'}(u_i) + \frac{d \log \phi^{'}(u_i)}{d \log(u_i)}[\log(u_{i+1}) - \log(u_i)], \tag{4.25}$$

$$\log \phi^{'}(u_{i-1}) \geq \log \phi^{'}(u_i) + \frac{d \log \phi^{'}(u_i)}{d \log(u_i)}[\log(u_{i-1}) - \log(u_i)]. \tag{4.26}$$

So it gives

$$\frac{\log \frac{\phi^{'}(u_{i+1})}{\phi^{'}(u_i)}}{\log(u_i) - \log(u_{i+1})} \leq a^R(u_i) \leq \frac{\log \frac{\phi^{'}(u_{i-1})}{\phi^{'}(u_i)}}{\log(u_i) - \log(u_{i-1})}. \tag{4.27}$$

Using the fact that $x_s^t = \sum_{j=1}^J r_{sj} y_j^t$, $U_s^t = u(\sum_{j=1}^J r_{sj} y_j^t)$, and $\rho_a^t = \phi^{'}(\sum\limits_{s=1}^{S} \nu_{as} u(\sum_{j=1}^J r_{js} y_j^t))$, the above inequality can be rewritten as

$$\frac{\log \rho_{i+1} - \log \rho_i}{\log(u_i) - \log(u_{i+1})} \leq a^R(u_i) \leq \frac{\log \rho_{i-1} - \log \rho_i}{\log(u_i) - \log(u_{i-1})}. \tag{4.28}$$

Sufficiency:

Once the observed portfolio choices satisfy the conditions in the theorem, the above ratio conditions give an upper and a lower bound on risk and ambiguity aversion at each level of contingent consumption. Thus these ratio conditions are necessary for decreasing (or increasing) absolute (or relative) risk and ambiguity aversion.

However, such conditions are also sufficient conditions. If the Afriat numbers satisfy the ratio conditions, I can construct increasing and concave functions $u(x)$ and $\phi(u)$ that exhibit decreasing (or increasing) absolute (or relative) risk and ambiguity aversion. In this proof, I construct a smooth ambiguity utility displaying decreasing absolute risk aversion and decreasing relative ambiguity aversion, and other cases can be constructed similarly.

Pick a set of numbers r_i^A and a_i^R satisfying the *Afriat Ratio Condition i* and *Afriat Relative Ratio Condition ii* respectively.

Define the function

$$u(x) = \int_0^x M(x)dx = \int_0^x exp[\log M(x)]dx, \tag{4.29}$$

where

$$\log M(x) = \max_i \{\log M_i - r_i^A (x - x_i)\}. \tag{4.30}$$

Define the function

$$\phi(u) = \int_0^u \rho(u)du = \int_0^u exp[\log \rho(u)]du, \tag{4.31}$$

where

$$\log \rho(u) = \max_i \{\log \rho_i - a_i^R (\log u - \log u_i)\}. \tag{4.32}$$

Then we will show the constructed overall utility index $U(z)$ will rationalize the data $(P^t, y^t, \nu^t, \mu^t)^{t=1,\dots,T}$, and exhibits decreasing absolute risk aversion and decreasing relative ambiguity aversion.

From the construction, the above functions are differentiable at x_i and u_i respectively, and we have

$$\frac{d \log M(x_i)}{dx} = \frac{M^{'}(x_i)}{M(x_i)} = -r_i^A, \tag{4.33}$$

$$\frac{d \log \rho(u_i)}{d \log u} = u_i \frac{\rho^{'}(u_i)}{\rho(u_i)} = -a_i^R. \tag{4.34}$$

The first derivative of $U(\boldsymbol{y})$ at y_j is $\sum_{a=1}^{A} \mu_a \rho(u_a) \sum_{s=1}^{S} \nu_{as} M(x_s) r_{sj}$.

It can be proved that $\rho(u_i) = \rho_i$ and $M(x_i) = M_i$.

Since the Afriat numbers satisfy the conditions $\sum_{a=1}^{A} \mu_a \rho_a \sum_{s=1}^{S} \nu_{as} M_s r_{sj} = P_j$ in the Lemma 5, then the following condition holds:

$$\sum_{a=1}^{A} \mu_a \rho(u_a) \sum_{s=1}^{S} \nu_{as} M(x_s) r_{sj} = P_j. \tag{4.35}$$

The second derivative of $U(\boldsymbol{y})$ at y_j is < 0.

Thus $U(\boldsymbol{y})$ is a concave function. So the satisfaction of the first-order conditions is a sufficient condition for the observed choice to solve the maximization problem.

Finally, the absolute risk aversion at y_j is given by

$$r^A(x_s) = -\frac{u^{''}(x_s)}{u^{'}(x_s)} = r_s^A, \tag{4.36}$$

which is a decreasing sequence by construction.

And the relative ambiguity aversion is given by

$$a^R(u_a) = -u_a \frac{\phi''(u_a)}{\phi'(u_a)} = a^R_a, \tag{4.37}$$

which is a decreasing sequence by construction.

□

Remark 33. It is not possible to aggregate consumers' preferences in order to bound the average risk and ambiguity aversions. This can be seen from equation (4.35): the marginal utility will contain both risk attitude M_s and ambiguity attitude ρ_a, and the number of equations is much less than the number of the unknowns. And it should be noted that the loss of aggregation is not due to incomplete markets: suppose fully complete Arrow securities (contingent on both risk and ambiguity states) are traded, it is still not possible to uniquely pin down ρ_a and M_{as} (in this case, state consumption is contingent on both risk and ambiguity states) from AS prices.

4.4 Conclusion

The revealed preference test is based on finite observations. The construction of strictly increasing and strictly concave functions rationalizing the observations is not unique, since only finite data are available. Kubler, Selden, and Wei (2014) give two demand function tests for the expected utility model, Dybvig and Polemarchakis (1981) give an argument on recovering the expected utility uniquely from asset demands, and their argument is extended in Chapter 3 to recover ambiguity preference. One interesting open question is what is the corresponding Slutsky condition on demand function for decreasing or increasing absolute (relative) risk and ambiguity aversion? Is it possible to uniquely identify such preference from demand functions?

Chapter 5

Risk and ambiguity aversion: empirical evidence

5.1 Introduction

Despite the importance of individual risk aversion and ambiguity aversion in determining individual decision making and equilibrium implications, the identification and estimation of individual risk aversion and ambiguity aversion have not been tackled until recently. And there is rare evidence on the shape of individual ambiguity preferences, except few evidence from either lab experiment or pure thought experiment using variants of Ellsberg's urns.

This chapter intends to investigate systematically the nature of household ambiguity preference using household survey data on consumption and portfolio choice and on stock return expectation. To derive explicitly the approximate solution, I assume that both relative risk aversion and relative ambiguity aversion are constant. One important desirable property of such a utility function is that the preference representation is invariant to the measurement unit of the risk aversion index, which does not hold generally under smooth ambiguity model. I also assume that the distribution of the risky asset return is log-normal, and that households have ambiguity on the mean with second-order belief being normally distributed. The assumption of ambiguity on the mean is assumed to achieve identification, and is justified by empirical evidence that the volatility of a stock return is more predictable than its mean.

The key theoretical result on identification is Proposition 11. It shows that if an individual's consumption and portfolio choice, as well as expectations of risky asset return, are observed, then individual time preference, relative risk aversion, and relative ambiguity aversion can be uniquely identified from a special panel dataset, where the individual in one period only holds riskfree asset, and in the other period holds both riskless and risky assets. However, the required panel data is very rare; instead I assume the same time preference across individuals, and use cross-section data to recover individual risk and ambiguity aversion (Corollary 7). The assumption of homogeneous time preference is justified in the empirical section.

Individual subjective expectations required in the identification argument are pinned down by the data from a household questionnaire and historical data from the Italy stock market. Although the effort to elicit individual belief quantitatively in household survey questionnaire began in the 1990s, there are few surveys that measure individual ambiguous belief. [1] Since I assume individual has no ambiguity on the variance of stock return, the variance is approximated by historical volatility of Italy stock return. In Bank of Italy Survey on Household Income and Wealth (SHIW), households were asked two questions about their expectations regarding stock market performance in Italy expressed on a probability scale, in 2008 and 2010 respectively.[2] Given the assumption on household expectation and ambiguous belief, the unconditional distribution of expected stock return would be log-normal, and its mean and variance can be recovered from data on two expectation questions. Then the ambiguity can be decomposed from the recovered variance given the historical stock return volatility, which is assumed to be known to households.

I examine whether constant relative risk aversion (CRRA) and constant relative ambiguity aversion (CRAA) are a good approximation, and test the ambiguous belief assumption against the subjective expectation hypothesis (in Savage's sense). CRRA and CRAA assumption puts strong restriction on observable choices: the saving rate out of wealth is independent of wealth

[1] Manski (2004) provides an extensive survey on using household expectation data over various events including stock market return.

[2] In the most recent wave SHIW2012, households are also asked to give their expectations on Italy stock market return, however, the questions changed and the results are not directly comparable with data in 2008 and 2010 waves. So in this chapter, I only use expectation data in 2008 and 2010, where the same questions were asked.

level, and the risky asset share out of saving is independent of saving level. These restrictions will be tested using SHIW2008 and SHIW2010 data. If households are not ambiguity averse, and have subjective expectation, the evidence from data that saving rate is invariant to wealth and risky asset share is invariant to saving suggests households are CRRA expected utility maximizers. Lemma 8 shows that the CRRA model implies an over-identification restriction, based on which I can distinguish the two models. Another way to distinguish these two models is to test whether the recovered relative ambiguity aversion is zero, since the expected utility model is a special case when the ambiguity aversion is zero.

Once I have confirmed that constant relative risk aversion and relative ambiguity aversion are a reasonable assumption, and recovered individual relative risk and ambiguity aversion from their consumption and portfolio choice, I can answer some interesting and important questions: will individual risk aversion and ambiguity aversion be correlated? how will individual risk aversion and ambiguity aversion be influenced by their characteristics, and how will individual risk aversion and ambiguity aversion influence their consumption and portfolio choice?

My empirical analysis is based on SHIW2008 and SHIW2010, a rotating panel data set, which contains detailed information on household socio-demographic characteristics, consumption expenditure, saving, portfolio allocation among various financial assets, and expectations of stock market performance. Firstly, I recover household belief regarding mean and variance of log-normal return from their answer to questions about stock market performance expectations. In contrast to the widely used homogeneous expectation assumption, household expectations display quite a lot of heterogeneity in terms of both mean and variance. This can happen when households have private information on stock market, or they process publicly available information in different ways. Household expectations are very pessimistic in the sense that the expected mean return is very low, and the expected simple excess return is barely positive. Even among households holding risky assets, a large proportion hold negative simple excess return expectation. The recovered household belief also reveals that households are subject to a lot of ambiguity, which is measured by the difference between recovered variance and historical volatility.

The analysis confirms that constant relative risk aversion and ambiguity aversion can be a good first approximation. I test the hypothesis using both cross-sectional regression and first difference panel regression. In the cross-sectional test, households' risky asset shares out of saving are hardly variant to saving level. The conclusion is both statistically and economically significant. The saving rate out of wealth is significantly related with wealth level even after controlling endogeneity of wealth and the effect of other variables, however, when I concentrate on the population expecting positive excess return, the effect of wealth on saving rate is not different from zero. The panel test based on first difference gives strong support for the predictions that the effect of saving on risky asset share and the effect of wealth on consumption rate are not significant.

The recovery of risk and ambiguity aversion is conducted first based on a small panel required in Proposition 11, assuming that the riskfree interest rate is invariant across two years. The recovered risk aversion and ambiguity aversion are quite heterogeneous across households. The relative risk aversion is much small than 1, and the relative ambiguity aversion is around 3 or larger. However, household time preferences are very homogeneous, and are consistent with existing micro evidence. Without loss of generality, I assume the same time preference across households, and recover household risk and ambiguity aversion from cross-sectional data. The picture from cross-sectional data is similar to that from panel data. t test cannot reject the hypothesis that both relative risk aversion and ambiguity aversion are significantly different from zero. The over-identification test rejects the CRRA expected utility model, in favor of the ambiguity model. Further analysis shows that household risk aversion and ambiguity aversion are not correlated. Household characteristics can hardly explain the variation of risk aversion and ambiguity aversion across households. Quantitatively, both risk aversion and ambiguity aversion have a significant effect on their consumption and portfolio choice; but ambiguity aversion mainly affects household portfolio rather than consumption choice.

Contributions and related literature

The most important contribution of this chapter is to simultaneously identify individual risk aversion and ambiguity aversion within a simple framework using household survey data. According to my knowledge, this is the first paper to use real data to recover household risk and ambiguity aversion. The difficulty lies in too many parameters for the model to identify, including individual first-order and second-order beliefs. This chapter solves the difficulty by imposing some reasonable parametric restrictions on underlying preference and belief and taking advantage of high quality household survey data.

This chapter is most closely related to the experimental approach to eliciting and estimating individual ambiguity aversion. Ahn, Choi, Gale, and Kariv (2014) perform a portfolio-choice experiment, where participants could choose three Arrow securities subject to a budget constraint. The payoff of the three Arrow securities is contingent on the realization of three states, where the probability of one state is objectively given, and the relative probabilities of the other states are ambiguous. Ahn, Choi, Gale, and Kariv (2014) estimate individual risk aversion and ambiguity aversion from their portfolio choice under kinked specification (α-Maxmin model) and smooth specification (smooth ambiguity aversion model) respectively. They also estimate a third parameter−pessimism, under a generalized kinked specification. Their main conclusion is that individual preferences exhibit considerable heterogeneity; a majority of subjects conforms to subjective expected utility hypothesis; most of the remaining subjects exhibit statistically significant ambiguity aversion or seeking and/or pessimism, but no subject displays extreme ambiguity aversion as supposed in the maxmin model; the estimated absolute risk aversion is much lower than the standard single-parameter estimates; ambiguity aversion and risk aversion are not correlated.

This chapter provides insight into the functional form of risk aversion and ambiguity aversion. The previous literature assumes a particular functional form and derives the implication of ambiguity aversion; however, the functional form is taken for granted, and has never been tested. This chapter provides empirical evidence on the shape of individual risk and ambiguity utility function. Constant relative risk aversion and relative ambiguity aversion can be a good approximation, and possess one desirable property-it is independent of the unit of risk aversion index, which does not hold generally under smooth ambiguity aversion model.

This chapter also contributes to testing the implication of ambiguity aversion, and distinguishing different models. The current literature mainly uses the experimental approach. Halevy (2007) exploits the tight association between ambiguity attitude and the reduction of compound lotteries axiom to design a lab experiment to compare the performance of subjective expected utility, maxmin expected utility, recursive non-expected utility and recursive expected utility. Participants are presented four urns, of which one represents pure risk, one represents pure ambiguity, and the other two represent *objective* compound lotteries. Their reservation values for these four urns are elicited through Becker-DeGroot-Marschak Becker, DeGroot, and Marschak (1964) mechanism. Different models will have strong prediction on the ranking of reservation values of these four urns. The result confirms the tight association between ambiguity neutrality and the reduction of compound lotteries. The population are heterogeneous, and there is no unique theory to explain the average decision maker. His result also reveals two choice patterns when participants show non-ambiguity neutrality and the reduction of compound lotteries is violated. Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010) design an experiment, where participants trade bond and three Arrow securities. The payoff of the three Arrow securities depends on the realization of three states, where the probability of one state is given, and the relative probabilities of the other two states are ambiguous. The equilibrium asset price is determined by a continuous open-book double auction. Ambiguity aversion will produce different implication for portfolio choice and equilibrium asset price compared to risk aversion. Specifically, under α-maxmin model, there exists an open set of price for which ambiguity averse subjects will not hold ambiguous assets, and the ranking of state price/probability ratios can be anomalous. Their experiment result confirms the prediction, and their result also supports the kinked ambiguity aversion model rather than smooth one. Using similar experimental design, Ahn, Choi, Gale, and Kariv (2014) present evidence in favor of kinked ambiguity aversion model in portfolio choice setting.

Another contribution of this chapter is to provide insight into how individual risk and ambiguity aversion are correlated with each other, how they are differentially influenced by individual characteristics, and how they influence individual consumption and portfolio choice. Dimmock, Kouwenberg, Mitchell, and Peijnenburg (2013) measure household ambiguity aversion from custom-designed questions based on Ellsberg urns in a large representative survey of U.S. households to test the effects of ambiguity aversion on household portfolio choice. They show that ambiguity aversion is negatively associated with stock market participation and with the fraction of wealth allocated to stocks, and the effect is large. Borghans, Golsteyn, Heckman, and Meijers (2009) design an experiment to examine the gender difference in risk aversion and ambiguity aversion. Participants are presented four urns filled with balls of two different colors, of which the first urn represents pure risk, and from the second to fourth urn, ambiguity is increased. Participants are asked to bet on one color and give the minimum price at which they would be willing to sell the bet. They measure ambiguity aversion by the difference in reservation prices between urn four and urn one. They show that woman are more risk-averse than men, and over an initial range, men reduce their valuation of ambiguous urns more than women, after that, men and women equally value marginal changes in ambiguity. They also shows that psychological characteristics account for some of the interpersonal variation in risk aversion, but not the difference in ambiguity aversion.

The chapter is related to recovering individual belief using expectation data on stock market performance. Hurd, Rooij, and Winter (2011) assume the stock return is log-normally distributed, and use stock market expectation data from Dutch household survey to estimate the mean and variance. They find that households' expectations are heterogeneous, and they are correlated with stock ownership. On average, stock market expectations are much more pessimistic about gains than the historical record of actual gains.

The rest of this chapter proceeds as follows. Section 5.2 presents a simple parametric portfolio choice model under the framework of smooth ambiguity aversion, where an explicit approximate solution is derived. Section 5.3 gives the key identification result, and sets out the empirical strategy. Section 5.4 describes the data I use. Section 5.5 reports and discusses the estimation and recovery results. Section 5.6 contains a robustness check of my results. Section 5.7 concludes and discusses possible future work. Appendix A establishes identifiability for the case relative risk aversion being larger than 1, and Appendix B derives the economic meaning of relative risk aversion and relative ambiguity aversion.

5.2 Economic model

5.2.1 Expected utility model

As in previous chapters, I consider a one-good two-period economy: period 0 and period 1, where uncertainty will be revealed in period 1. The consumer is endowed with initial wealth I and CRRA preference

$$u(c) = \frac{c^{1-\rho}}{1-\rho}, \tag{5.1}$$

where ρ is the relative risk aversion index. There is one good to consume and two assets for the consumer to save for tomorrow, of which one asset is risky, and the other is risk-free. Unlike in previous chapters, here I assume a continuum of states. The risky asset has a return factor $\tilde{\boldsymbol{\nu}}$ for each dollar with $\ln(\tilde{\boldsymbol{\nu}}) \sim N(\mu, \sigma^2)$, and the riskless asset has a return factor R for each dollar with $\ln(R) = r$. The consumer's problem is to decide how much to consume today and how much to save for tomorrow, and how much of saving is invested in the risky asset. I suppose the only risk is from the asset return, and abstract away from individual background risk−individual wealth is given. This will simplify the solution to individual decision problem.

Let c_0 be the first period consumption, and α be the share of saving $(I - c_0)$ invested in the risky asset, then the second-period consumption will be

$$c_1 = \Big(R + (\tilde{\boldsymbol{\nu}} - R)\alpha\Big)(I - c_0). \tag{5.2}$$

If individual knows exactly the distribution of the asset return, and maximizes his expected utility, then he will solve the following problem:

$$\begin{aligned} &\max_{\{c_0,\alpha\}} u(c_0) + \beta E_{\tilde{\boldsymbol{\nu}}} u(c_1) \\ &s.t.\ c_1 = \big(R + (\tilde{\boldsymbol{\nu}} - R)\alpha\big)(I - c_0). \end{aligned} \tag{5.3}$$

From log-normal approximation, see Campbell and Viceira (2002), the following is obtained:

$$\ln\Big(R + (\tilde{\boldsymbol{\nu}} - R)\alpha\Big) \approx r + \alpha\Big(\ln(\tilde{\boldsymbol{\nu}}) - \ln(R)\Big) + \alpha\frac{\sigma^2}{2} - \alpha^2\frac{\sigma^2}{2}. \tag{5.4}$$

Then second-period expected utility is

$$
\begin{aligned}
E_{\tilde{\boldsymbol{\nu}}} u(c_1) &= E \frac{\left[(I-c_0)\exp\left(r+\alpha\big(\ln(\tilde{\boldsymbol{\nu}})-\ln(R)\big)+\alpha\frac{\sigma^2}{2}-\alpha^2\frac{\sigma^2}{2}\right)\right]^{1-\rho}}{1-\rho} \\
&= \frac{(I-c_0)^{1-\rho}}{1-\rho}\exp\Big(C(\alpha)\Big)E_{\tilde{\boldsymbol{\nu}}}\exp\Big((1-\rho)\alpha\ln(\tilde{\boldsymbol{\nu}})\Big), \qquad (5.5)
\end{aligned}
$$

where

$$
C(\alpha) = (1-\rho)\big(r+\alpha\frac{\sigma^2}{2}-\alpha^2\frac{\sigma^2}{2}-\alpha r\big). \qquad (5.6)
$$

From the above assumption on asset returns, it follows

$$
(1-\rho)\alpha\ln(\tilde{\boldsymbol{\nu}}) \sim N\Big((1-\rho)\alpha\mu, (1-\rho)^2\alpha^2\sigma^2\Big). \qquad (5.7)
$$

For a normal random variable $\boldsymbol{z} \sim N(\mu, \sigma^2)$, $E\exp(\boldsymbol{z}) = \exp\big(\mu + \frac{\sigma^2}{2}\big)$. Thus,

$$
\begin{aligned}
E_{\tilde{\boldsymbol{\nu}}} u(c_1) &= \frac{(I-c_0)^{1-\rho}}{1-\rho}\exp\Big(C(\alpha)\Big)\exp\left((1-\rho)\alpha\mu+\alpha^2\frac{(1-\rho)^2\sigma^2}{2}\right) \\
&= \frac{(I-c_0)^{1-\rho}}{1-\rho}\exp\left((1-\rho)\big(r+\alpha(\mu-r)+\alpha\frac{\sigma^2}{2}-\alpha^2\frac{\rho\sigma^2}{2}\big)\right). \qquad (5.8)
\end{aligned}
$$

Then, the individual's optimization problem becomes

$$
\max_{\{c_0,\alpha\}} \frac{c_0^{1-\rho}}{1-\rho}+\beta\frac{(I-c_0)^{1-\rho}}{1-\rho}\exp\left((1-\rho)\big(r+\alpha(\mu-r)+\alpha\frac{\sigma^2}{2}-\alpha^2\frac{\rho\sigma^2}{2}\big)\right). \qquad (5.9)
$$

So the optimal solution is:

$$
\alpha = \frac{\mu-r+\frac{\sigma^2}{2}}{\rho\sigma^2}, \qquad (5.10)
$$

$$
c_0 = \kappa w, \text{where } \kappa = \left[1+\beta^{\frac{1}{\rho}}\exp\left(\frac{(1-\rho)}{\rho}\big(r+\frac{(\mu-r+\frac{\sigma^2}{2})^2}{2\rho\sigma^2}\big)\right)\right]^{-1}. \qquad (5.11)
$$

Remark 34. Equations (5.10) and (5.11) generate interesting comparative statics:

1. Risky asset demand is positively related to the simple excess return

(or equity premium) $\mu - r + \frac{\sigma^2}{2}$, and inversely related to the risk aversion ρ.

2. First period consumption is inversely related to time preference β, and positively related to risk aversion ρ.

5.2.2 Smooth ambiguity model

However, in reality individuals are not so sophisticated at knowing the exact probability distribution of future events, in fact, they are ambiguous. This is especially true in the financial market−even complicated statistical models cannot predict the return of assets. I suppose investors know the variance σ^2, but have ambiguity over the mean of the risky asset μ, i.e. μ is a random variable $\tilde{\boldsymbol{\mu}}$ for individuals. The assumption of ambiguity about the mean is justified: firstly, the mean of asset returns is more difficult to predict than the variance, see Epstein and Schneider (2010); secondly, as shown in Chapter 3, ambiguity about the mean is one sufficient condition for identifying ambiguity aversion index uniquely; thirdly, under ambiguity about the mean, the unconditional distribution of the asset return can be easily derived, which enables me to recover individual belief. I assume that the mean return $\tilde{\boldsymbol{\mu}}$ is normally distributed, i.e. $\tilde{\boldsymbol{\mu}} \sim N(\theta, \sigma_0^2)$, where σ_0^2 represents the ambiguity, i.e. the larger σ_0^2, the more ambiguous is the individual. Then the compound distribution of the risky asset return $\tilde{\boldsymbol{\nu}}$ would be also log-normal, i.e. $\ln\tilde{\boldsymbol{\nu}} \sim N(\theta, \sigma^2 + \sigma_0^2)$.

I assume that the individual is endowed with smooth ambiguity preference, as in Klibanoff, Marinacci, and Mukerji (2005). Let the monotone and concave function

$$\phi(u) = \frac{u^{1-A}}{1-A}, \tag{5.12}$$

represent ambiguity aversion, where concavity of $\phi(u)$ indicates how ambiguity averse the individual is. As in the risk aversion case, define $-\frac{\phi''}{\phi'}$ as absolute ambiguity aversion, and $-u\frac{\phi''}{\phi'}$ as relative ambiguity aversion. Smooth ambiguity preference models the individual's decision as he evaluates the consumption bundle c by $\phi^{-1}\Bigg[E\phi\Bigg(Eu(c)\Bigg)\Bigg]$, where

$$\phi^{-1} = \big((1-A)\phi\big)^{\frac{1}{1-A}}. \tag{5.13}$$

The ambiguity aversion function $\phi(u)$ is defined on positive u, which assumes $\rho < 1$. For $\rho > 1$, the ambiguity aversion is defined to be $\phi(u) = -\frac{(-u)^{1+A}}{1+A}$, the corresponding result will be relegated to the appendix A.

The following lemma shows how demand for the risky asset depends on its equity premium $E_{\tilde{\boldsymbol{\mu}}}E_{\tilde{\boldsymbol{\nu}}}\tilde{\boldsymbol{\nu}} - R$, which is evaluated by the unconditional (or average) probability.

Lemma 6. *If an individual is both risk averse and ambiguity averse, then the necessary and sufficient condition for him to hold positive amount of the risky asset is* $E_{\tilde{\boldsymbol{\mu}}}E_{\tilde{\boldsymbol{\nu}}}\tilde{\boldsymbol{\nu}} - R > 0$.

Proof. Without loss of generality, I assume the individual has one unit of wealth to invest between the risk-free asset and the risky asset. The return would be $R + (\tilde{\boldsymbol{\nu}} - R)\alpha$. Maximizing $\phi^{-1}\Big\{E_{\tilde{\boldsymbol{\mu}}}\Big[\phi\Big(E_{\tilde{\boldsymbol{\nu}}}u\big(R + (\tilde{\boldsymbol{\nu}} - R)\alpha\big)\Big)\Big]\Big\}$ is equivalent to maximizing $E_{\tilde{\boldsymbol{\mu}}}\Big[\phi\Big(E_{\tilde{\boldsymbol{\nu}}}u\big(R + (\tilde{\boldsymbol{\nu}} - R)\alpha\big)\Big)\Big]$, which is an increasingly monotone transformation.

Take a quadratic Taylor expansion at $\alpha = 0$, the following obtains

$$E_{\tilde{\boldsymbol{\mu}}}\phi\Big(E_{\tilde{\boldsymbol{\nu}}}u\big(R + (\tilde{\boldsymbol{\nu}} - R)\alpha\big)\Big) = \phi\Big(u(R)\Big) + \phi'\Big(u(R)\Big)u'(R)(E_{\tilde{\boldsymbol{\mu}}}E_{\tilde{\boldsymbol{\nu}}}\tilde{\boldsymbol{\nu}} - R)\alpha$$
$$+ \frac{\phi''\Big(u(R)\Big)u'^2(R)E_{\tilde{\boldsymbol{\mu}}}(E_{\tilde{\boldsymbol{\nu}}}\tilde{\boldsymbol{\nu}} - R)^2 + \phi'\Big(u(R)\Big)u''(R)E_{\tilde{\boldsymbol{\mu}}}E_{\tilde{\boldsymbol{\nu}}}(\tilde{\boldsymbol{\nu}} - R)^2}{2}\alpha^2. \tag{5.14}$$

For the individual to have an incentive to hold positive amount of the risky asset, i.e. $E_{\tilde{\boldsymbol{\mu}}}\phi\Big(E_{\tilde{\boldsymbol{\nu}}}u\big(R + (\tilde{\boldsymbol{\nu}} - R)\alpha\big)\Big) - \phi\Big(u(R)\Big) > 0$, the following condition is needed

$$E_{\tilde{\boldsymbol{\mu}}}E_{\tilde{\boldsymbol{\nu}}}\tilde{\boldsymbol{\nu}} - R > -\Big[\frac{\phi''\Big(u(R)\Big)}{\phi'\Big(u(R)\Big)}u'(R)\frac{E_{\tilde{\boldsymbol{\mu}}}(E_{\tilde{\boldsymbol{\nu}}}\tilde{\boldsymbol{\nu}} - R)^2}{2} + \frac{u''(R)}{u'(R)}\frac{E_{\tilde{\boldsymbol{\mu}}}E_{\tilde{\boldsymbol{\nu}}}(\tilde{\boldsymbol{\nu}} - R)^2}{2}\Big]\alpha, \tag{5.15}$$

where the right hand side is positive given that the individual is both risk averse and ambiguity averse. $\square$

Remark 35. Equation (5.15) shows how much equity premium an individual requires for him to hold α units of the risky asset, whose return distribution is ambiguous. The required equity premium depends on individual risk aversion, ambiguity aversion, ambiguity of asset return, and the magnitude of α. A risk loving and/or ambiguity loving individual would like to hold some positive amount of the risky asset even if his subjective equity premium is negative.

Remark 36. Equation (5.15) also shows that generally, the required equity premium is not independent of the unit of risk aversion index u, the meaning of which is not clear. However, when the relative ambiguity aversion i.e. $-\frac{\phi''\left(u(R)\right)}{\phi'\left(u(R)\right)}u'(R)$ on the right hand side of (5.15), is constant, such dependence disappears, and the result is much easier to interpret.

Under above model specification, the consumer will solve the following maximization problem:

$$\begin{aligned} &\max_{\{c_0,\alpha\}} u(c_0) + \beta\phi^{-1}\Big(E_{\tilde{\boldsymbol{\mu}}}\phi\big(E_{\tilde{\boldsymbol{\nu}}}u(c_1)\big)\Big) \\ &s.t.\ c_1 = \Big(R + (\tilde{\boldsymbol{\nu}} - R)\alpha\Big)(I - c_0). \end{aligned} \tag{5.16}$$

Substitute the budget constraint into the objective function, then the problem becomes

$$\max_{\{c_0,\alpha\}} u(c_0) + \beta\phi^{-1}\Big\{E_{\tilde{\boldsymbol{\mu}}}\phi\Big[E_{\tilde{\boldsymbol{\nu}}}u\Big(\big(R + (\tilde{\boldsymbol{\nu}} - R)\alpha\big)(I - c_0)\Big)\Big]\Big\}. \tag{5.17}$$

The log-normal approximation of portfolio return would be

$$\ln\Big(R+(\tilde{\boldsymbol{\nu}}-R)\alpha\Big) \approx r+\alpha\Big(\ln(\tilde{\boldsymbol{\nu}})-\ln(R)\Big)+\alpha\frac{\sigma^2+\sigma_0^2}{2}-\alpha^2\frac{\sigma^2+\sigma_0^2}{2}. \tag{5.18}$$

Then the expected second-period utility conditional on $\tilde{\boldsymbol{\mu}}$ is

$$\begin{aligned} E_{\tilde{\boldsymbol{\nu}}}u(c_1) &= E_{\tilde{\boldsymbol{\nu}}}u\Big(\big(R + (\tilde{\nu} - R)\alpha\big)(I - c_0)\Big) \\ &= \frac{(I - c_0)^{1-\rho}}{1-\rho}\exp\Big(F(\alpha)\Big)\exp\Big((1-\rho)\alpha(\mu - r)\Big), \end{aligned} \tag{5.19}$$

where

$$F(\alpha) = (1-\rho)\big(r + \alpha\frac{\sigma^2+\sigma_0^2}{2} - \alpha^2\frac{\rho\sigma^2+\sigma_0^2}{2}\big). \tag{5.20}$$

Determination of α

Given ϕ, it can be shown that

$$E_{\tilde{\boldsymbol{\mu}}}\phi\Big(E_{\tilde{\boldsymbol{\nu}}}u(c_1)\Big) = \frac{\Big(\frac{(I-c_0)^{1-\rho}}{1-\rho}\Big)^{1-A}\exp\Big(G(\alpha)\Big)}{1-A} E_{\tilde{\boldsymbol{\mu}}}\exp\Big((1-A)(1-\rho)\alpha\tilde{\boldsymbol{\mu}}\Big), \tag{5.21}$$

where

$$G(\alpha) = (1-A)(1-\rho)\Big(r + \alpha\frac{\sigma^2+\sigma_0^2}{2} - \alpha^2\frac{\rho\sigma^2+\sigma_0^2}{2} - \alpha r\Big). \tag{5.22}$$

Given the assumption that $\tilde{\boldsymbol{\mu}} \sim N(\theta, \sigma_0^2)$,

$$(1-A)(1-\rho)\alpha\tilde{\boldsymbol{\mu}} \sim N\Big((1-A)(1-\rho)\alpha\theta, (1-A)^2(1-\rho)^2\alpha^2\sigma_0^2\Big). \tag{5.23}$$

It gives

$$E_{\tilde{\boldsymbol{\mu}}}\exp\Big((1-A)(1-\rho)\alpha\tilde{\boldsymbol{\mu}}\Big) = \exp\Big((1-A)(1-\rho)\alpha\theta + \alpha^2\frac{(1-A)^2(1-\rho)^2\sigma_0^2}{2}\Big). \tag{5.24}$$

Substitute equation (5.24) into equation (5.21), the following is obtained:

$$E_{\tilde{\boldsymbol{\mu}}}\phi\Big(E_{\tilde{\boldsymbol{\nu}}}u(c_1)\Big) = \frac{\Big(\frac{(I-c_0)^{1-\rho}}{1-\rho}\Big)^{1-A}\exp\Big((1-A)(1-\rho)H(\alpha)\Big)}{1-A}, \tag{5.25}$$

where

$$H(\alpha) = r + \alpha(\theta - r + \frac{\sigma^2+\sigma_0^2}{2}) - \alpha^2\frac{\rho\sigma^2+\sigma_0^2}{2} + \alpha^2\frac{(1-A)(1-\rho)\sigma_0^2}{2}. \tag{5.26}$$

So the second-period utility is

$$\phi^{-1}\Big[E_{\tilde{\boldsymbol{\mu}}}\phi\Big(E_{\tilde{\boldsymbol{\nu}}}u(c_1)\Big)\Big] = \frac{(I-c_0)^{1-\rho}}{1-\rho}\exp\Big((1-\rho)K(\alpha)\Big), \tag{5.27}$$

where

$$K(\alpha) = r + \alpha(\theta - r) + \alpha\frac{\sigma^2+\sigma_0^2}{2} - \alpha^2\frac{\rho\sigma^2+\sigma_0^2}{2} + \alpha^2\frac{(1-A)(1-\rho)\sigma_0^2}{2}. \tag{5.28}$$

In the individual maximization problem, the choice of risky asset share α is uncorrelated with the choice of first-period consumption c_0. So without loss of generality, to solve the following maximization problem will determine his portfolio choice:

$$\max_{\{\alpha\}} \ \phi^{-1}\left[E_{\tilde{\boldsymbol{\mu}}}\phi\left(E_{\tilde{\boldsymbol{\nu}}}u(c_1)\right)\right]. \tag{5.29}$$

First order condition:

$$\theta - r + \frac{\sigma^2 + \sigma_0^2}{2} - \left((\rho\sigma^2 + \sigma_0^2) - (1-A)(1-\rho)\sigma_0^2\right)\alpha = 0. \tag{5.30}$$

Then the optimal solution is

$$\alpha = \frac{\theta - r + \frac{\sigma^2+\sigma_0^2}{2}}{\rho(\sigma^2+\sigma_0^2) + (1-\rho)A\sigma_0^2}. \tag{5.31}$$

Determination of c_0

$$\max_{\{c_0,\alpha\}} \ u(c_0) + \beta\phi^{-1}\left[E_{\tilde{\boldsymbol{\mu}}}\phi\left(E_{\tilde{\boldsymbol{\nu}}}u(c_1)\right)\right]. \tag{5.32}$$

F.O.C w.r.t c_0,

$$c_0^{-\rho} = \beta(I - c_0)^{-\rho}\exp\left((1-\rho)K(\alpha)\right). \tag{5.33}$$

Therefore,

$$c_0 = \kappa I, \tag{5.34}$$

where

$$\kappa = \left\{1 + \beta^{\frac{1}{\rho}}\exp\left[\frac{1-\rho}{\rho}\left(r + \frac{(\theta - r + \frac{\sigma^2+\sigma_0^2}{2})^2}{2\left(\rho(\sigma^2+\sigma_0^2) + (1-\rho)A\sigma_0^2\right)}\right)\right]\right\}^{-1}. \tag{5.35}$$

Remark 37. Note when the individual is ambiguity neutral, i.e. $A = 0$, the optimal solution (5.31) and (5.35) will coincide with the optimal choice of CRRA expected utility maximizer using the compound distribution. Smooth ambiguity aversion model characterizes individual ambiguity aversion by relaxing the reduction of first-order and second-order belief when they evaluate ambiguous asset.

Remark 38. Equations (5.31) and (5.35) generate interesting comparative statics:

1. Demand for the risky asset is positively related to equity premium $\theta - r + \frac{\sigma^2+\sigma_0^2}{2}$, and inversely related to ambiguity aversion A; however, unlike in pure risk case, the effect of risk aversion ρ on risky asset demand is indeterminate, depending on the magnitude of σ^2, σ_0^2, and A.

2. The first period consumption is inversely related to time preference β, and positively related to ambiguity aversion A; unlike in pure risk case, the effect of risk aversion ρ on first period consumption is indeterminate.

5.3 Econometric strategy

5.3.1 Identification of preference parameters

Suppose individual belief information can be elicited out: θ−the mean of normal distribution of risky asset mean return, σ_0−the variance of normal distribution of risky asset mean return and σ−the variance of risky asset return, I will show that all parameters of individual preference: time preference β, risk aversion ρ, and ambiguity aversion A can be identified uniquely. At first glance, there are two equations that determine α, κ, and three variables β, ρ, and A, and it seems these parameters cannot be uniquely identified. Actually it can if there is panel data, where under the same riskfree asset return, in one period he only holds risky free asset, and in the other period he holds both riskless and risky assets. Proposition 11 gives the key identification result.

Proposition 11. *Assume the individual has constant relative risk aversion and constant relative ambiguity aversion preference. Assume the subjective distribution of the risky asset return is log-normal with ambiguous mean being normally distributed. Suppose*

1. *at time s, with asset returns $(\tilde{\boldsymbol{\nu}}_s, R)$, the individual has consumption rate κ^s, and only invests in the riskfree asset;*

2. *at time t, with asset return $(\tilde{\boldsymbol{\nu}}_t, R)$, the individual has consumption rate κ^t, and invests α^t of saving in the risky asset.*

Then individual time preference β, relative risk aversion ρ, and relative ambiguity aversion A can be uniquely identified.

Proof. Step 1—Identification of relative risk aversion ρ

Given the risky asset share equation

$$\alpha = \frac{\theta - r + \frac{\sigma^2+\sigma_0^2}{2}}{\rho(\sigma^2+\sigma_0^2) + (1-\rho)A\sigma_0^2}, \tag{5.36}$$

and the consumption rate equation

$$\kappa = \left\{1+\beta^{\frac{1}{\rho}} \exp\left[\frac{(1-\rho)}{\rho}\left(r+\frac{(\theta - r + \frac{\sigma^2+\sigma_0^2}{2})^2}{2\big(\rho(\sigma^2+\sigma_0^2) + (1-\rho)A\sigma_0^2\big)}\right)\right]\right\}^{-1}, \tag{5.37}$$

the consumption rate equation can be expressed as

$$\kappa = \left\{1+\beta^{\frac{1}{\rho}} \exp\left[\frac{(1-\rho)}{\rho}\left(r+\frac{\alpha(\theta - r + \frac{\sigma^2+\sigma_0^2}{2})}{2}\right)\right]\right\}^{-1}. \tag{5.38}$$

Suppose, in period s, the individual only demands the riskfree asset, and his consumption and asset choice are observed, with consumption rate being κ^s, then the following is obtained:

$$\kappa^s = \left[1+\beta^{\frac{1}{\rho}} \exp\left(\frac{(1-\rho)}{\rho} r\right)\right]^{-1}. \tag{5.39}$$

So

$$\beta^{\frac{1}{\rho}} \exp\left(\frac{(1-\rho)}{\rho} r\right) = \frac{1-\kappa^s}{\kappa^s}. \tag{5.40}$$

Suppose, in period t, the individual demands both the riskless and risky assets, with the risky asset share being α^t, then,

$$\kappa^t = \left[1+\frac{1-\kappa^s}{\kappa^s} \exp\left(\frac{(1-\rho)}{\rho} \frac{\alpha^t(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2})}{2}\right)\right]^{-1}. \tag{5.41}$$

In the above equation (5.41), the only unknown parameter is risk aversion ρ, and it can be recovered as

$$\rho = \frac{\frac{\alpha^t(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2})}{2}}{\ln\frac{1-\kappa^t}{\kappa^t} - \ln\frac{1-\kappa^s}{\kappa^s} + \frac{\alpha^t(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2})}{2}}. \tag{5.42}$$

Step 2–Identification of time preference β

When demand for the risky asset is zero, from consumption rate equation (5.39), β can be identified as

$$\beta = (\frac{1-\kappa^s}{\kappa^s})^\rho \exp\Big((\rho-1)r\Big), \tag{5.43}$$

once ρ has been identified.

Step 3–Identification of relative ambiguity aversion A

The risky asset share equation (5.36) gives

$$\alpha^t(1-\rho)A\sigma_{0t}^2 = (\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2}) - \alpha^t\rho(\sigma_t^2+\sigma_{0t}^2). \tag{5.44}$$

So ambiguity aversion can be identified as

$$A = \frac{(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2}) - \alpha^t\rho(\sigma_t^2+\sigma_{0t}^2)}{\alpha^t(1-\rho)\sigma_{0t}^2}, \tag{5.45}$$

where the right hand side has been known (either observable or identified in Step 1 and Step 2). □

Remark 39. The identification argument in Chapter 3 can recover individual ambiguity preference nonparametrically, but requires observing the whole demand functions. The identification Proposition 11 is a parametric special case of Chapter 3, and requires a few observations to recover individual preference, which enables me use household survey data to achieve identification.

When I apply the above identification strategy to data, one problem I encounter is that even when I have a panel data, the observation of both zero and nonzero demand for the risky asset under an invariant interest rate is rare. In this case, instead of identifying individual time preference, I assume homogeneous time preference across households and take a value consistent with micro-data evidence, and I focus on recovering the other two parameters–relative risk aversion ρ and relative ambiguity aversion A, which are more interesting. I will make a clear justification for such an assumption in the following empirical part.

Corollary 7. *Assume the individual has constant relative risk aversion and constant relative ambiguity aversion preference. Assume the subjective distribution of the risky asset return is log-normal with ambiguous mean being normally distributed. Suppose at time t, with asset return $(\tilde{\boldsymbol{\nu}}_t, R)$, the individual has consumption rate κ^t, and invests α^t of saving in the risky asset. If individual time preference β is known, then individual relative risk aversion and relative ambiguity aversion can be uniquely identified as*

$$\rho = \frac{\ln\beta + r + \frac{\alpha^t(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2})}{2}}{\ln\frac{1-\kappa^t}{\kappa^t} + r + \frac{\alpha^t(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2})}{2}}, \tag{5.46}$$

$$A = \frac{(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2}) - \alpha^t\rho(\sigma_t^2 + \sigma_{0t}^2)}{\alpha^t(1-\rho)\sigma_{0t}^2}. \tag{5.47}$$

5.3.2 Recovery of individual belief

In the model, I assume that the return of the risky asset $\tilde{\boldsymbol{\nu}}$ for each dollar follows a log-normal distribution, i.e. $\ln(\tilde{\boldsymbol{\nu}}) \sim N(\mu, \sigma^2)$, and that the individual is ambiguous on the mean of expected return $\tilde{\boldsymbol{\mu}}$, which is assumed to be a normal random variable i.e. $\tilde{\boldsymbol{\mu}} \sim N(\theta, \sigma_0^2)$. Then the unconditional distribution of risky asset return $\tilde{\boldsymbol{\nu}}$ is also log-normal $\ln(\tilde{\boldsymbol{\nu}}) \sim N(\theta, \sigma^2 + \sigma_0^2)$. Individual belief parameters θ, σ^2, and σ_0^2 are recovered in the following way.

Individuals are assumed to have access to historical stock market data, and base their belief on this data. I assume the individual has no ambiguity on the variance of the risky asset return, so I calculate the variance of historical stock returns (after taking ln transformation), and use it to approximate σ^2.

In the survey data I use, individuals answer questions about probabilities of future stock market returns. Future stock market returns are assumed to be log-normal, and individuals perceive the distribution of future stock market returns with its unconditional (or average) probability. Then both θ and $\sigma^2 + \sigma_0^2$ can be estimated from the data.

Specifically, belief information is elicited out from two questions in the questionnaire:

a. On a scale from 0 to 100, what is the likelihood that if you invest in the Italian stock market today it will yield a profit in a year's time?

b. (If you give a figure for question a) What is the likelihood the investment will earn more than 10%?

Denote belief from question a) and question b) by p_a and p_b. Before I use the households' reported belief data to recover the subjective distribution of the risky asset return, I need to check the quality of the belief data, since such data are more subjective than the data of wealth variables, and it relies on whether households can think in a probabilistic way. I will do the following check:

Consistency check: If households think in a probabilistic way, then both p_a and p_b should satisfy the weak consistency condition as probability i.e. $p_a \geq p_b$. A more demanding requirement is the strict consistency condition i.e. $p_a > p_b$.

As will be seen in the next sections, the belief data passes this test very well, and I will use these data to recover individual belief in two different ways.

Face value approach

Firstly, I take the household expectation data at its face value, and believe that households truthfully report what they think without any error. Such face value approach would be objected by the argument that the survey does not give any incentive to household to truthfully reveal their belief; however, households do not have incentive to truthfully report the value of other variables like consumption, portfolio and income either. Just as the reported household consumption, portfolio and income are useful for our understanding their decision making, I believe that the reported household expectation data contains information of their perception over the stock return distribution. In the analysis of the next section, I do the consistency check, and find that more than 90% households report consistent belief. So it indicates that households think these questions carefully, rather than give random answers. Given p_a and p_b, then I have the following recovery result:

Lemma 7. *Suppose the individual subjective distribution of the risky asset return is log-normal with the ambiguous mean being normally distributed.*

Given p_a and p_b, then individual belief can be uniquely recovered as

$$\theta = \frac{\ln 1.1}{\Phi^{-1}(1-p_a) - \Phi^{-1}(1-p_b)}\Phi^{-1}(1-p_a), \tag{5.48}$$

$$\sigma_0^2 = \frac{\ln 1.1}{\Phi^{-1}(1-p_b) - \Phi^{-1}(1-p_a)} - \sigma^2. \tag{5.49}$$

Proof. Based on the questions about stock expectation return, I have

$$p_a = P(\tilde{\boldsymbol{\nu}} \geq 1), \tag{5.50}$$

$$p_b = P(\tilde{\boldsymbol{\nu}} \geq 1.1). \tag{5.51}$$

From the above two equations, I have

$$p_a = P(\ln \tilde{\boldsymbol{\nu}} \geq \ln 1), \tag{5.52}$$

$$p_b = P(\ln \tilde{\boldsymbol{\nu}} \geq \ln 1.1), \tag{5.53}$$

i.e.

$$p_a = P(\frac{\ln \tilde{\boldsymbol{\nu}} - \theta}{\sigma^2 + \sigma_0^2} \geq \frac{-\theta}{\sigma^2 + \sigma_0^2}), \tag{5.54}$$

$$p_b = P(\frac{\ln \tilde{\boldsymbol{\nu}} - \theta}{\sigma^2 + \sigma_0^2} \geq \frac{\ln 1.1 - \theta}{\sigma^2 + \sigma_0^2}). \tag{5.55}$$

Since $\frac{\ln \tilde{\boldsymbol{\nu}} - \theta}{\sigma^2 + \sigma_0^2}$ is a standard normal variable, the equations (5.54) and (5.55) give the result in the lemma, where Φ is the cumulative distribution function (CDF) of the standard normal distribution. □

Market data approach

In the above face value approach, I do not allow any report or measurement error, and do not put any restriction on household second order belief- i.e. their expectation on the mean return, except assuming it's normally distributed, so households' expected stock equity premium can be far away from historical realization. However, households probably answer these questions by rounding rather than exactly, and it's possible that there are measurement errors. Besides, although households do not know exactly the probability distribution of stock market return, usually they do have a rough idea about the equity premium from stock market, i.e. they do not totally ignore the information of market data on the expected equity return. So my second approach to household belief data is to allow error in the belief data, and put restrictions on household second order belief such that the perceived mean equity return matches the market data.

Specifically, I assume the true values p_a^* and p_b^* are proportional to p_a and p_b by a common factor λ, i.e. $p_a^*{=}\lambda p_a$ and $p_b^*{=}\lambda p_b$. So the magnitude of λ reflects how much the reported value deviates from its true value. Then if $\lambda = 1$, the reported belief data is exact without error; if $\lambda > 1$, households report their belief in a conservative way. I assume the error factor λ is the same across p_a and p_b, one motivation is if p_a and p_b satisfy consistency conditions, then λp_a and λp_b will also satisfy these conditions.

I assume households are ambiguous on the mean of the log-normal distribution, and their perceived unconditional mean equity return equals the historical realization, which is denoted by $\hat{E}$. Under these assumptions, knowing p_a, p_b and $\hat{E}$ suffices to recover the three parameters θ, σ_0^2 and λ, since the following three equations hold:

$$\lambda p_a = P\left(\frac{\ln \tilde{\boldsymbol{\nu}} - \theta}{\sigma^2 + \sigma_0^2} \geq \frac{-\theta}{\sigma^2 + \sigma_0^2}\right), \tag{5.56}$$

$$\lambda p_b = P\left(\frac{\ln \tilde{\boldsymbol{\nu}} - \theta}{\sigma^2 + \sigma_0^2} \geq \frac{\ln 1.1 - \theta}{\sigma^2 + \sigma_0^2}\right), \tag{5.57}$$

$$\hat{E} = \theta + \frac{\sigma^2 + \sigma_0^2}{2}. \tag{5.58}$$

The unknown can be expressed implicitly by the following equations:

$$\theta = \frac{\ln 1.1}{\Phi^{-1}(1-\lambda p_a) - \Phi^{-1}(1-\lambda p_b)}\Phi^{-1}(1-\lambda p_a), \tag{5.59}$$

$$\sigma_0^2 = \frac{\ln 1.1}{\Phi^{-1}(1-\lambda p_b) - \Phi^{-1}(1-\lambda p_a)} - \sigma^2, \tag{5.60}$$

$$\frac{\ln 1.1}{2} = \hat{E}\Phi^{-1}(1-\lambda p_b) - (\hat{E} - \ln 1.1)\Phi^{-1}(1-\lambda p_a). \tag{5.61}$$

Since Φ^{-1} function is nonlinear with respect to λ, I can not express the solution explicitly. I will use numerical methods to get the solution.

5.3.3 Testing constant relative risk and ambiguity aversion

One testable implication of constant relative risk aversion and constant relative ambiguity aversion is that the consumption rate out of wealth is independent of wealth level, and the portfolio share out of saving is independent of saving level. To test the constant relative risk aversion and constant relative ambiguity aversion assumption, I run the following regression:

$$\ln \kappa_{it} = a_1 + b_1 x_{it} + d_1 \ln I_{it} + \mu_{it}, \tag{5.62}$$

$$\ln \alpha_{it} = a_2 + b_2 x_{it} + d_2 \ln s_{it} + \eta_{it}, \tag{5.63}$$

where $\ln \kappa_{it}$ is the ln of saving rate, $\ln \alpha_{it}$ is the ln of risky asset share, x_{it} is a vector of individual specific control variables including their age, gender, education etc, $\ln I_{it}$ is individual wealth level, and $\ln s_{it}$ is individual saving level. The parameters of interest are d_1 and d_2, which, according to the theory, should be 0.

However, if there exist unobservable individual characteristics υ_i and ν_i affecting consumption and portfolio choice, then the true regressions would be

$$\ln \kappa_{it} = a_1 + b_1 x_{it} + d_1 \ln I_{it} + \upsilon_i + \mu_{it}, \tag{5.64}$$

$$\ln \alpha_{it} = a_2 + b_2 x_{it} + d_2 \ln s_{it} + \nu_i + \eta_{it}. \tag{5.65}$$

If risk aversion or ambiguity aversion are related to wealth/saving level, but not observable, then it will be part of μ_i and η_i, which will cause an endogeneity problem. One approach to dealing with the endogeneity problem is to find an instrument, which is correlated with wealth but not correlated with unobservable preference, and run a two stage regression or use the GMM method to estimate the model.

If the unobservable preference heterogeneity is fixed over time, another approach is to use panel data to difference the fixed effect. Under this approach, I need to control the variation of belief since across periods individuals will change their belief, and hence change their consumption and portfolio choice, even though they are not related to wealth level. To test the assumption of constant relative risk aversion and constant relative ambiguity aversion, I first difference the fixed effect, then run the following regression:

$$\triangle \ln \kappa_{it} = a_1 + b_1 \triangle x_{it} + d_1 \triangle \ln w_{it} + \triangle \mu_{it}, \tag{5.66}$$

$$\triangle \ln \alpha_{it} = a_2 + b_2 \triangle x_{it} + d_2 \triangle \ln s_{it} + \triangle \eta_{it}. \tag{5.67}$$

Since I have a rotating panel from SHIW2008 and SHIW2010, I will test constant relative risk and ambiguity aversion using both the instrumental variable method and the first difference method.

5.3.4 Distinguishing two models

Testing over-identification restrictions

Note that the fact that the consumption rate is invariant to wealth level and the risky asset share is invariant to saving level cannot distinguish the CRRA expected utility model from the smooth ambiguity model in this chapter. One can interpret this as evidence that individuals have ambiguity on the distribution of risky asset return, and are ambiguity averse; alternatively, one can interpret it as evidence that individuals are expected utility maximizers with CRRA utility functions. Then how could the two models be distinguished from each other based on the data? The following lemma shows that CRRA preference puts over-identification restrictions on recovered risk aversion ρ, which is a means of distinguishing between these two models.

Lemma 8. *Assume the individual has constant relative risk aversion, and the risky asset return is log-normally distributed. Suppose*

1. *at time s, with asset returns $(\tilde{\boldsymbol{\nu}}_s, R)$, the individual has consumption rate κ^s, and only invests in the riskfree asset;*

2. *at time t, with asset return $(\tilde{\boldsymbol{\nu}}_t, R)$, the individual has consumption rate κ^t, and invests α^t of saving in the risky asset.*

Then relative risk aversion ρ will be over-identified.

Proof. I only sketch the proof, since it is similar to the proof of Proposition 11.

From Section 5.2, under expected utility theory, a CRRA expected utility maximizer will make the following choice:

$$\alpha = \frac{\mu - r + \frac{\sigma^2}{2}}{\rho\sigma^2}, \tag{5.68}$$

$$\kappa = \left\{1 + \beta^{\frac{1}{\rho}} \exp\left[\frac{(1-\rho)}{\rho}\left(r + \frac{(\mu - r + \frac{\sigma^2}{2})^2}{2\rho\sigma^2}\right)\right]\right\}^{-1}. \tag{5.69}$$

In period s, observing $\kappa^s = \left[1 + \beta^{\frac{1}{\rho}} \exp\left(\frac{(1-\rho)}{\rho} r\right)\right]^{-1}$,

$$\beta^{\frac{1}{\rho}} \exp\left(\frac{(1-\rho)}{\rho} r\right) = \frac{1-\kappa^s}{\kappa^s}. \tag{5.70}$$

In period t, observing κ^t and α^t,

$$\kappa^t = \left\{1 + \beta^{\frac{1}{\rho}} \exp\left[\frac{(1-\rho)}{\rho}\left(r + \alpha^t \frac{(\mu_t - r + \frac{\sigma_t^2}{2})}{2}\right)\right]\right\}^{-1}. \tag{5.71}$$

From κ^t, risk aversion can be recovered as

$$\rho_\kappa = \frac{\frac{\alpha^t(\mu_t - r + \frac{\sigma_t^2}{2})}{2}}{\ln\frac{1-\kappa^t}{\kappa^t} - \ln\frac{1-\kappa^s}{\kappa^s} + \frac{\alpha^t(\mu_t - r + \frac{\sigma_t^2}{2})}{2}}. \tag{5.72}$$

From α^t, risk aversion can be recovered as

$$\rho_\alpha = \frac{\mu_t - r + \frac{\sigma_t^2}{2}}{\alpha^t\sigma_t^2}. \tag{5.73}$$

□

If subjective expected utility theory holds, then relative risk aversion ρ identified from α and κ, i.e. ρ_α and ρ_κ should be the same. The testable implication is to test the equality. When I use cross-sectional data rather than panel data to identify relative risk aversion, I assume homogeneous time preference across individuals, and take a value consistent with micro evidence.

Corollary 8. *Assume the individual has constant relative risk aversion, and the risky asset return is log-normally distributed. Suppose at time*

t, with asset return $(\tilde{\boldsymbol{\nu}}_t, R)$, the individual has consumption rate κ^t, and invests α^t of saving in the risky asset. If individual time preference β is known, then risk aversion ρ will be over-identified as

$$\rho_\alpha = \frac{\mu_t - r + \frac{\sigma_t^2}{2}}{\alpha^t \sigma_t^2}, \tag{5.74}$$

and

$$\rho_\kappa = \frac{\ln \beta + r + \frac{\alpha^t(\mu_t - r + \frac{\sigma_t^2}{2})}{2}}{\ln \frac{1-\kappa^t}{\kappa^t} + r + \frac{\alpha^t(\mu_t - r + \frac{\sigma_t^2}{2})}{2}}. \tag{5.75}$$

Testing ambiguity aversion restriction

As remarked, when the individual shows ambiguity neutrality, i.e. $A = 0$, he will act as a subjective expected utility maximizer using compound (or unconditional) probability. So another way to distinguish between the subjective expected utility model and the smooth ambiguity model is to test whether the recovered ambiguity aversion A equals zero or not. If the recovered ambiguity aversion A is significantly different from 0, this will cast doubt on the subjective expected utility theory.

5.3.5 Factors affecting risk aversion and ambiguity aversion

What factors affect individual risk attitude and ambiguity attitude? Can individual characteristics like age, gender, education, marital status etc, account for large heterogeneity in risk aversion and ambiguity aversion across individuals? To answer this question, I will run the following regressions:

$$\rho_i = \alpha_1 + \beta_1 z_i + \epsilon_i, \tag{5.76}$$

$$A_i = \alpha_2 + \beta_2 z_i + \xi_i, \tag{5.77}$$

where z_i is a vector of control variables including gender, age, number of household members, education, and wealth.

5.4 Data description

I use the data from the Survey on Household Income and Wealth (SHIW) collected by the Bank of Italy every two years, which contains detailed information on the socioeconomic characteristics, income, saving, and portfolios of more than 8000 Italian households. SHIW is a rotating panel with around half of households interviewed in the next wave. This analysis uses two waves of the survey: SHIW2008 and SHIW2010. These two waves are used mainly because they contain questions about household expectations about stock market performance which are consistent and comparable. SHIW2008 and SHIW2010 cover 7977 and 7951 households respectively, and the number of panel households (interviewed in both waves) is 4621.[3]

5.4.1 Construction of key variables

All wealth variables in the survey refer to the household as a whole and are self-reported, end of year, market value measured in Euro.

Consumption: refers to household expenditure on nondurable goods over the whole year. Due to the nature of durable goods, I do not consider durable goods in most of the analysis; however, a robustness check including durable goods will be performed.

Riskfree asset: includes deposits (bank accounts, certificates of deposit, repos, and post office savings certificates) and Italian government securities (T-bills, T-certificates, T-bonds, zero coupons, and other).

Risky financial asset: includes bonds issued by Italian firms or banks, Italian investment funds (money market or liquidity funds, bond funds, balanced funds, equity funds, index funds etc), shares of listed or unlisted companies and equity in partnerships, managed portfolios, foreign securities issued by non-residents, loans to cooperatives, and other financial assets (options, futures, royalties etc). [4]

Saving: is the sum of riskfree assets and risky financial assets.

[3]In SHIW2010, based on randomization (birth year of household head being odd or even), only half of the households are asked to answer these questions.

[4]In the analysis, I exclude the value of business equity and the value of real estate from risk financial assets. First of all, the inclusion of business equity and real estate would be inconsistent with household belief data, which is concerned with the return from the stock market. Secondly, both business equity and real estate are more than a standard asset, and the associated risk with them is unclear.

Household wealth: is the sum of consumption expenditure and saving.

5.4.2 Individual belief

In SHIW2008 and SHIW2010, all interviewed households are asked to express their expectations about investment returns from the Italian stock market:

a. On a scale from 0 to 100, what is the likelihood that if you invest in the Italian stock market today it will yield a profit in a year's time?

b. (If you give a figure for question a) What is the likelihood the investment will earn more than 10%?

These two questions can determine the whole distribution of a two-parameter probability distribution, which is assumed to be log-normal in this chapter. However, not all interviewed households answered these questions and answered in a consistent way. In SHIW2008, 2245 out of 7977 (28.1%) households answer both questions; however, among households holding risky assets, 706 out of 1291 (54.7%) households answer both questions. In SHIW2010, 1347 out of 4135−after randomization (32.6%) households answer both questions; among households holding risky assets, 405 out of 762−after randomization (53.1%) households answer both questions. Households holding risky assets are more likely to report their expectations about stock market performance; this is probably because such households are more rich and well-educated, are accustomed to probabilistic thinking, and track the stock market more often.

In the analysis, I only keep strictly consistent answers, where the probability of the first question is larger than that of the second, i.e. $p_a > p_b$.[5] In SHIW2008, 1734 out of 7977 (21.7%) households report consistent beliefs; among households holding risky assets, the ratio increases to 40.9% (528 out of 1291). In SHIW2010, 1120 out of 4135 (27.1%) households give reasonable answers; among households holding risky assets, the ratio increases to 45.1% (344 out of 762). Among 4621 panel households, only 315 households report consistent expectations in both interviews.

[5] By strictly consistent I mean $p_a > p_b$; by weakly consistent I mean $p_a >= p_b$.

Consistency check: although relatively few households answer both questions, however, conditional on reporting their beliefs in both questions, most households give consistent answers: in SHIW2008, 1734 out of 2245 (77.2%) households show strict consistency, and 2079 out of 2245 (92.6%) households show weak consistency; in SHIW2010, 1120 out of 1347 (83.1%) households show strict consistency, and 1292 out of 1347 (95.9%) households show weak consistency; and this is also true among households holding risky assets.

To use household expectation data, I make following adjustment: replace extreme values with less extreme ones. In the case $p_a = 100$, households express they are very confident on positive return, we replace it with $p_a = 99$. In the case $p_b = 0$, households don't think it's possible to have more than 10% return, I replace it with $p_b = 1$. I expect such a small adjustment would not incur a loss of generality.

5.4.3 Descriptive analysis

Table 5.1 presents the descriptive statistics of the key variables and the sample size, where the upper panel includes all sample households, and the lower panel is based on households holding risky assets. Some patterns in the data deserve mentioning.

First of all, across groups, households holding risky assets have a higher level of consumption, saving and wealth, but a lower consumption rate than average households in both 2008 and 2010. This fact is consistent with other existing evidence on household consumption and saving behavior, which shows households with risky assets are generally more rich. In terms of their expectations, in both 2008 and 2010, households holding risky assets are more optimistic, i.e. higher mean of both p_a and p_b than average households, and their expectations are more dispersed, i.e. a higher standard deviation of both p_a and p_b.

Table 5.1: Descriptive statistics

	SHIW 2008			SHIW 2010		
	All sample households			All sample households		
	mean	*Std.Dev.*	*Observation*	*mean*	*Std.Dev.*	*Observation*
Consumption	22.190	11.642	7977	23.676	13.763	7951
Riskfree asset	17.332	60.351	7977	17.376	53.471	7951
Saving	24.386	78.302	7977	27.999	94.199	7951
Wealth	46.576	82.646	7977	51.675	100.325	7951
Consumption rate	0.701	0.257	7977	0.695	0.261	7951
Consistent belief	-	-	1734	-	-	1120
p_a	25.654	21.785	1734	23.498	20.544	1120
p_b	8.329	13.176	1734	6.457	10.796	1120
	Holding risky asset			**Holding risky asset**		
	mean	*Std.Dev.*	*Observation*	*mean*	*Std.Dev.*	*Observation*
Consumption	31.538	16.328	1291	33.731	18.678	1484
Riskfree asset	35.917	84.757	1291	33.896	58.153	1484
Risky asset	43.592	102.905	1291	56.916	157.472	1484
Saving	79.509	139.129	1291	90.813	176.365	1484
Wealth	111.047	145.006	1291	124.544	184.382	1484
Consumption rate	0.416	0.206	1291	0.404	0.195	1484
Risky asset share	0.535	0.281	1291	0.567	0.276	1484
Consistent belief	-	-	528	-	-	344
p_a	31.169	23.945	528	29.401	23.054	344
p_b	11.258	15.934	528	8.407	12.735	344

Note: The unit of consumption, riskfree asset, risky asset, saving, and wealth is 1000 Euros. Consumption refers to expenditure in nondurable goods. Risky assets refer to risky financial assets, i.e. firm or bank bonds, investment funds, stock shares and other financial assets, excluding business equity and real estate. Consistent belief refers to belief with strict consistency, i.e. $p_a > p_b$. In SHIW 2010, based on head of household's birth year being even or odd, only half of the interviewed households are asked to report their expectation of stock market performance.

Secondly, across time, the levels of consumption, saving and wealth, and the proportion of households holding risky assets, have increased from 2008 to 2010; however, the consumption rate (for both general households and households with risky assets) and the risky asset share (for households with risky assets) keep relatively stable across these two years. Before 2008, when the financial crisis occurred, more than 25% of Italian households had risky assets; however, following financial crisis, only 16.2% of households held risky assets in 2008, so the effect of the financial crisis is obvious. This increases to 18.7% in 2010, indicating better economic prospects. In terms of household belief, people are more pessimistic in 2010, i.e. there is a lower mean of both p_a and p_b, which holds for both the general population and the one with risky asset. This is probably because households form their expectations based on their previous experience.

5.5 Result and interpretation

5.5.1 Recovering individual belief

Distribution of individual belief

I assume that households have no ambiguity about the variance of stock returns, and they forecast the variance based on historical data. In the analysis, I take the variance of stock log-normal return to be σ^2=0.02. For the return of the riskfree asset, let $R = 0.983$.[6] As mentioned in Section 5.4, I make small adjustments to household belief data with extreme values to make the computation possible: in SHIW2008, 13 reported $p_a = 1.00$ are replaced with $p_a = 0.99$, 39 reported $p_a = 0.01$ are replaced with $p_a = 0.02$, 587 reported $p_b = 0$ are replaced with $p_b = 0.01$; in SHIW2010, 8 reported $p_a = 1.00$ are replaced with $p_a = 0.99$, 29 reported $p_a = 0.01$ are replaced with $p_a = 0.02$, and 445 reported $p_b = 0$ are replaced with $p_b = 0.01$. Table 5.2 reports the summary statistics of recovered household beliefs, i.e. the mean and variance of the log-normal distribution, both in 2008 and in 2010, where the upper half includes all the sample households who report consistent beliefs, and the lower half concentrates exclusively on households holding positive risky financial assets.

[6] The data on Italy stock and bond return are based on the analysis in Dimson, Marsh, and Staunton (2011).

Table 5.2: Distribution of recovered household belief

	Individual belief in 2008 All sample households			Individual belief in 2010 All sample households		
	θ	$\sigma_0^2+\sigma^2$	$E\tilde{\nu}$	θ	$\sigma_0^2+\sigma^2$	$E\tilde{\nu}$
5%	-0.718	0.046	-0.543	-0.718	0.041	-0.543
10%	-0.336	0.058	-0.205	-0.383	0.053	-0.267
25%	-0.230	0.085	-0.124	-0.230	0.079	-0.160
50%	-0.117	0.119	-0.041	-0.117	0.123	-0.071
75%	-0.028	0.217	0.017	-0.028	0.217	-0.001
95%	0.048	0.350	0.117	0.037	0.350	0.091
Min	-1.481	0.021	-1.031	-1.781	0.021	-1.178
Max	0.253	0.901	0.550	0.278	1.207	0.403
Mean	-0.151	0.157	-0.072	-0.167	0.155	-0.090
Std. Dev.	0.195	0.107	0.162	0.204	0.108	0.164
Observation	1734	1734	1734	1120	1120	1120
	Holding risky asset			**Holding risky asset**		
	θ	$\sigma_0^2+\sigma^2$	$E\tilde{\nu}$	θ	$\sigma_0^2+\sigma^2$	$E\tilde{\nu}$
5%	-0.718	0.041	-0.543	-0.403	0.041	-0.295
10%	-0.336	0.053	-0.205	-0.336	0.046	-0.205
25%	-0.158	0.074	-0.074	-0.182	0.064	-0.074
50%	-0.077	0.119	-0.022	-0.077	0.098	-0.022
75%	0.000	0.217	0.039	0.000	0.182	0.028
95%	0.095	0.352	0.186	0.064	0.350	0.126
Min	-1.481	0.030	-1.031	-0.832	0.020	-0.550
Max	0.253	0.901	0.403	0.141	0.564	0.325
Mean	-0.119	0.154	-0.042	-0.119	0.137	-0.051
Std. Dev.	0.197	0.113	0.168	0.171	0.094	0.140
Observation	528	528	528	344	344	344

Note: I only include samples with strictly consistent belief, i.e. $p_a > p_b$. Simple return $E\tilde{\nu}$ is equal to $\theta + \frac{\sigma_0^2+\sigma^2}{2}$. Risky assets refer to risky financial assets-i.e. firm or bank bonds, investment funds, stock shares and other financial assets, excluding business equity and real estate.

I find that, first of all, household expectations are quite heterogeneous.

For the general population, in 2008 (2010), the perceived mean θ of log-normal return varies from -1.487 (-1.781) to 0.253 (0.278), the perceived variance $\sigma_0^2 + \sigma^2$ ranges from 0.021 (0.020) to 0.901 (1.207), and the perceived simple excess return goes from -1.043 (-1.189) to 0.550 (0.391) across households. Such heterogeneity holds among households holding risky assets. This finding casts doubt on the homogeneous expectation hypothesis usually assumed in finance and macroeconomic modeling. It also has important implication for preference aggregation under uncertainty.

Secondly, household expectations are very pessimistic in terms of both perceived mean θ and simple return $E\tilde{\boldsymbol{\nu}}$. Across all sample households, in both periods, the mean θ of unconditional log-normal distribution is negative and more than half of households expect θ would be negative. This also holds with regard to the simple return $E\tilde{\boldsymbol{\nu}}$. Households holding risky assets are more optimistic than average households, but there is pessimism also among this group. Such pessimism is possibly influenced by the economic situation, as the interviews were done following the financial crisis. One important feature is that among households holding positive risky assets, more than half believe that the simple excess return from stock markets would be negative. From Lemma 6, it is known that these households cannot be both risk averse and ambiguity averse. This indicates that a large proportion of households are risk loving and/or ambiguity loving.

Thirdly, households are subject to much ambiguity. The perceived variance $\sigma_0^2 + \sigma^2$ is strictly larger than historical stock return volatility. Under my model, one part of the variance, σ_0^2, represents individual ambiguity, reflecting individual subjective uncertainty over the mean of stock returns. From Table 5.2, it can be seen that household ambiguity is large compared with realized stock return volatility. One important reason for ambiguity is informational constraint, i.e. households have limited information about how the stock market works, or they have difficulty understanding information available to them.

The reported result on recovered household expectations offers one way to understand household financial decisions and the stock market puzzle. There is a large literature about the fact that only a small proportion of households hold risky assets, which is a puzzle since historically stock returns have generated a very high equity premium−see Haliassos and Bertaut (1995). One standard approach to solving such a puzzle is to consider entry and transaction costs in the stock market—see Vissing-Jorgensen (2002). However, the relevance of participation costs for holding risky assets is controversial, since the measurable participation cost is very small compared with the historical stock return. The finding in this chapter uncovers one alternative explanation: household expectations are too pessimistic, and they are very ambiguous about stock returns. As will be seen in later analysis, household expectation has significant effect on households' stock market participation and risky asset holdings.

Factors influencing individual belief

What can account for such large variation in household expectation? Table 5.3 and Table 5.4 show the OLS regression of household belief (θ in columns (1) and (3) and $\sigma_0^2+\sigma^2$ in columns (2) and (4)) on household characteristics for 2008 and 2010 respectively.

First of all, the mean θ is closely related to wealth level (after ln transformation), age and gender. Wealthy households expect a higher level of the mean of log-normal return, while households with older household heads have lower expectations. Interestingly, male household heads display more optimistic expectations about stock returns, which holds for the general population and for households with risky assets. In 2008, more educated households showed more optimism.

Second, the large variation of ambiguity cannot be explained by observable household characteristics. In 2008, male household heads showed less ambiguity, and more educated households were less ambiguous; however, such an effect is not significant in 2010 survey. Other variables like wealth level, age, marital status and region do not have any significant effect.

Table 5.3: OLS regression of household belief SHIW 2008

	Household belief in 2008			
	All HH		**HH w. risky asset**	
	(1)	(2)	(3)	(4)
logw2008	0.025***	-0.005	0.021*	-0.006
	(0.007)	(0.004)	(0.012)	(0.007)
No.household members	-0.007	-0.002	-0.002	-0.001
	(0.006)	(0.003)	(0.009)	(0.006)
Age	-0.001**	0.0002	-0.003***	0.0004
	(0.001)	(0.0003)	(0.001)	(0.0006)
No.income earner	0.014*	-0.006	0.023**	-0.009
	(0.008)	(0.004)	(0.011)	(0.007)
Male	0.029***	-0.014**	0.07***	-0.025*
	(0.011)	(0.006)	(0.03)	(0.014)
Married	0.015	-0.000	0.035	-0.015
	(0.016)	(0.009)	(0.031)	(0.017)
Separated	0.012	-0.0004	0.037	-0.009
	(0.023)	(0.013)	(0.050)	(0.027)
Widowed	0.043*	-0.019	0.049	-0.017
	(0.023)	(0.012)	(0.049)	(0.025)
Primary school	0.111*	-0.035	0.339***	-0.199***
	(0.061)	(0.028)	(0.039)	(0.022)
Secondary school	0.106*	-0.037	0.349***	-0.214***
	(0.061)	(0.028)	(0.039)	(0.021)
Uni.or more	0.137**	-0.039	0.368***	-0.204***
	(0.062)	(0.029)	(0.043)	(0.024)
Employee	-0.001	0.004	-0.420	0.017
	(0.014)	(0.008)	(0.026)	(0.016)
Self-employed	0.012	-0.0003	-0.001	0.011
	(0.016)	(0.009)	(0.028)	(0.018)
Live in North	-0.015	-0.009	0.017	0.008
	(0.015)	(0.008)	(0.038)	(0.018)
Live in Center	0.003	-0.003	0.008	0.020
	(0.016)	(0.009)	(0.040)	(0.019)
Intercept	-0.494***	0.273***	-0.691***	0.449***
	(0.092)	(0.048)	(0.136)	(0.073)
N	1734	1734	528	528
R^2	0.031	0.012	0.070	0.041

Note: In columns (1) and (3), the left-hand side variable is θ; in columns (2) and (4), the left-hand side variable is $\sigma_0^2 + \sigma^2$. Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

Table 5.4: OLS regression of household belief SHIW 2010

	Household belief in 2010			
	All HH		**HH w. risky asset**	
	(1)	(2)	(3)	(4)
logw2010	0.045***	-0.017***	0.036***	-0.010
	(0.008)	(0.005)	(0.014)	(0.008)
No.household members	0.001	-0.003	0.0009	-0.002
	(0.008)	(0.004)	(0.014)	(0.007)
Age	-0.001*	-0.0000	-0.001	-0.002
	(0.0007)	(0.004)	(0.001)	(0.006)
No.income earner	-0.005	0.0003	-0.009	0.006
	(0.009)	(0.005)	(0.015)	(0.008)
Male	0.048***	-0.012	0.051**	-0.008
	(0.014)	(0.008)	(0.022)	(0.011)
Married	-0.029	0.023*	-0.059**	0.028
	(0.022)	(0.012)	(0.030)	(0.018)
Separated	-0.020	0.012	-0.046	0.042*
	(0.028)	(0.013)	(0.042)	(0.022)
Widowed	-0.020	0.026*	-0.062	0.037
	(0.029)	(0.015)	(0.047)	(0.027)
Primary school	-0.043	-0.005	-0.037	0.018
	(0.058)	(0.020)	(0.050)	(0.030)
Secondary school	-0.039	0.0003	0.001	0.005
	(0.056)	(0.019)	(0.023)	(0.012)
Uni.or more	-0.025	-0.006	(omitted)	(omitted)
	(0.059)	(0.021)	()	()
Employee	0.0007	-0.005	0.031	-0.003
	(0.017)	(0.009)	(0.033)	(0.017)
Self-employed	0.004	-0.012	0.045	-0.028*
	(0.019)	(0.011)	(0.033)	(0.017)
Live in North	0.007	-0.009	0.004	-0.020
	(0.020)	(0.011)	(0.035)	(0.026)
Live in Center	0.017	-0.010	0.003	-0.029
	(0.023)	(0.013)	(0.037)	(0.028)
Intercept	-0.547***	0.346***	-0.467***	0.237**
	(0.097)	(0.049)	(0.156)	(0.093)
N	1120	1120	344	344
R^2	0.060	0.033	0.088	0.049

Note: In columns (1) and (3), the left-hand side variable is θ; in columns (2) and (4), the left-hand side variable is $\sigma_0^2 + \sigma^2$. Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

Financial market participation

How will household expectations be related to their decision to hold risky assets? Table 5.5 reports the probit regression of stock market participation decisions on their belief and observable characteristics.[7]

I first run the regression controlling for wealth level only, since existing literature documents that household wealth level has a large impact on the likelihood of their holding risky assets. Regression result in columns (1) and (3) confirms such conclusion, and such effect is both economically and statistically significant.

In columns (2) and (4), I control for the effect of household expectations and other variables. Wealth level , number of household members and living region have a significant effect: higher wealth level and living in the north or center are positively related to stock market participation, while the number of household members is negatively related to the participation decision. In 2008, household expectations (both mean and variance) have a significant positive effect on their participation. The positive effect of variance is surprising at first glance. One interpretation is that what matters for household financial decisions is the simple return, which equals to $\theta + \frac{\sigma_0^2+\sigma^2}{2}$, and households would like to hold risky assets if the return can compensate for its riskiness and ambiguity. However, the effect of household belief is not significant in 2010.

In the above probit regression, one concern is the endogeneity of the wealth variable. In IV estimation not reported here, I instrument household wealth with their net annual income, the null hypothesis of exogeneity is not rejected, and the above conclusion still holds.

[7] Marital status variables do not have any significant effect. Since it is not the variable of my interest, it will not be reported in the below regression tables. However, the reported result is after controlling for marital status effect.

Table 5.5: Probit regression for market participation

	Participation in 2008		**Participation in 2010**	
	(1)	(2)	(3)	(4)
logw2008	1.198***	1.117***	-	-
	(0.030)	(0.088)		
logw2010	-	-	1.217***	1.245***
			(0.029)	(0.104)
Theta	-	0.810***	-	0.576
		(0.283)		(0.395)
Sigma	-	1.210**	-	0.528
		(0.513)		(0.728)
No.household members	-	-0.118***	-	-0.129**
		(0.045)		(0.060)
Age	-	-0.008**	-	-0.004
		(0.004)		(0.006)
No.income earner	-	0.035	-	-0.017
		(0.060)		(0.076)
Male	-	0.146*	-	0.097
		(0.88)		(0.107)
Primary school	-	0.772	-	3.721***
		(0.614)		(0.238)
Secondary	-	0.948	-	3.676***
		(0.605)		(0.223)
Uni.or more	-	0.986	-	3.768***
		(0.610)		(0.257)
Employee	-	0.036	-	0.218
		(0.108)		(0.148)
Self-employed	-	-0.130	-	0.107
		(0.134)		(0.176)
Live in North	-	0.497***	-	0.555***
		(0.127)		(0.173)
Live in Center	-	0.640***	-	0.430***
		(0.139)		(0.191)
Intercept	-13.764***	-13.337***	-13.931***	-18.064***
	(0.324)	(1.055)	(0.316)	(1.126)
N	7977	1734	7951	1120
Pseudo R^2	0.270	0.357	0.339	0.343

Note: Participation is defined by positive amount of risky financial assets. Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

5.5.2 Testing constant relative risk aversion and ambiguity aversion

In this section, I test the reasonableness of constant relative risk aversion and relative ambiguity aversion assumptions. One testable implication is that: risky asset share out of saving is independent of saving level, and saving rate out of wealth is independent of wealth level. I use two approaches to test this implication: cross-sectional test and panel first-difference test. To control for endogeneity in the cross-sectional regression, I run a GMM regression, using net annual income and/or value of durable goods to instrument wealth.

Table 5.6 and Table 5.7 report the results on how the risky asset share is related to saving using data from 2008 and 2010 respectively. In columns (1), (3) and (5), I run OLS without controlling for endogeneity caused by saving level, in columns (2), (4) and (6), I instrument household saving with household net annual income and/or value of durable goods. The sample in columns (1) and (2) includes households with risky assets, in columns (3) and (4) includes households with risky assets and reporting their beliefs, and in columns (5) and (6) includes households with risky assets and positive excess return expectation. The dependent variable is the log of risky asset share, and the independent variables include log of household saving and other controls. The parameter of interest is the coefficient of log saving, which should be zero according to the model. In 2008, column (1) in Table 5.6 shows that the risky asset share is negatively related to saving level, which can be explained by increasing risk aversion or increasing ambiguity aversion; however, the effect is economically insignificant. After controlling for endogeneity, the effect becomes statistically insignificant, as shown in column (2). In columns (3) and (4), I control for the effect of household expectations, the regression without considering endogeneity cannot reject the null hypothesis that the coefficient is zero, but such conclusion changes once I control for endogeneity. In columns (5) and (6), I concentrate on the sample with positive excess return expectations, which is most relevant to our recovery. The coefficient of saving is not different from zero, and the GMM C statistic shows that there is no serious endogeneity among this sample.

Table 5.6: Cross-sectional test of risky asset share invariant to saving SHIW 2008

	Risky asset share 2008					
	(1)	(2)	(3)	(4)	(5)	(6)
logsaving2008	-0.044*	0.069	0.006	0.185**	-0.025	0.082
	(0.025)	(0.058)	(0.038)	(0.090)	(0.054)	(0.102)
Theta	-	-	0.501**	0.433**	-0.206	-0.213
			(0.196)	(0.197)	(0.538)	(0.517)
Sigma	-	-	1.143***	1.120***	0.852**	0.870**
			(0.335)	(0.333)	(0.359)	(0.348)
No.household members	-0.034	-0.025	-0.030	-0.028	-0.049	-0.037
	(0.030)	(0.029)	(0.045)	(0.043)	(0.059)	(0.056)
Age	-0.001	-0.004	-0.002	-0.008	-0.001	-0.005
	(0.003)	(0.003)	(0.005)	(0.005)	(0.006)	(0.006)
No.income earner	0.009	-0.015	0.045	0.009	0.080	0.055
	(0.035)	(0.036)	(0.051)	(0.051)	(0.070)	(0.068)
Male	0.017	-0.010	0.050	0.005	-0.156	-0.194
	(0.057)	(0.059)	(0.101)	(0.108)	(0.118)	(0.120)
Secondary	0.093	0.017	0.004	-0.072	-0.315**	-0.322**
	(0.078)	(0.086)	(0.145)	(0.146)	(0.143)	(0.141)
Uni.or more	-0.051	-0.084	-0.109	-0.285	-0.250	-0.320*
	(0.097)	(0.124)	(0.164)	(0.195)	(0.158)	(0.170)
Employee	-0.070	-0.075	-0.078	-0.106	-0.042	-0.070
	(0.070)	(0.071)	(0.104)	(0.108)	(0.135)	(0.132)
Self-employed	-0.019	-0.062	-0.049	-0.142	0.102	0.032
	(0.082)	(0.085)	(0.127)	(0.127)	(0.152)	(0.151)
Live in North	0.211**	0.194**	-0.057	-0.140	-0.148	-0.173
	(0.084)	(0.088)	(0.128)	(0.143)	(0.178)	(0.173)
Live in Center	0.283***	0.298***	-0.091	-0.134	-0.311	-0.299
	(0.088)	(0.091)	(0.135)	(0.142)	(0.204)	(0.197)
Intercept	-0.556*	-1.477***	-0.777*	-2.114***	0.142	-0.759
	(0.291)	(0.501)	(0.432)	(0.733)	(0.587)	0.895
N	1291	1291	528	528	259	259
R^2	0.017		0.026		0.065	0.044
p-value of Hansen's J test	-	0.571	-	-	-	0.470
p-value of GMM C statistic	-	0.031	-	0.032	-	0.246

Note: Risky assets refer to risky financial assets-i.e. firm or bank bonds, investment funds, stock shares and other financial assets, excluding business equity and real estate. Saving is the sum of riskfree assets and risky financial assets. Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

Table 5.7: Cross-sectional test of risky asset share invariant to saving SHIW 2010

	Risky asset share 2010					
	(1)	(2)	(3)	(4)	(5)	(6)
logsaving2010	-0.005	0.093**	-0.065	0.025	-0.097	0.113
	(0.022)	(0.039)	(0.053)	(0.104)	(0.068)	(0.168)
Theta	-	-	1.111**	1.021**	2.683**	2.614**
			(0.501)	(0.471)	(1.159)	(1.073)
Sigma	-	-	0.801	0.767	-0.216	-0.048
			(0.714)	(0.679)	(0.864)	(0.769)
No.household members	0.028	0.027	0.131*	0.133**	0.081	0.113
	(0.028)	(0.028)	(0.068)	(0.066)	(0.074)	(0.076)
Age	0.002	-0.001	-0.004	-0.006	0.007	0.002
	(0.002)	(0.003)	(0.006)	(0.005)	(0.007)	(0.008)
No.income earner	-0.056*	-0.063**	-0.086	-0.098	-0.074	-0.109
	(0.032)	(0.032)	(0.070)	(0.071)	(0.078)	(0.082)
Male	-0.093**	-0.099**	-0.095	-0.105	-0.073	-0.145
	(0.045)	(0.045)	(0.117)	(0.114)	(0.159)	(0.173)
Secondary	0.038	-0.031	-0.011	-0.062	0.206	0.067
	(0.073)	(0.076)	(0.230)	(0.214)	(0.301)	(0.297)
Uni.or more	0.027	-0.095	0.145	0.058	0.321	0.106
	(0.081)	(0.090)	(0.233)	(0.223)	(0.311)	(0.344)
Employee	-0.075	-0.081	-0.306**	-0.308	0.044	0.037
	(0.064)	(0.064)	(0.151)	(0.148)	(0.178)	(0.182)
Self-employed	-0.046	-0.075	-0.221	-0.241	0.124	0.053
	(0.069)	(0.070)	(0.159)	(0.158)	(0.192)	(0.191)
Live in North	0.109*	0.085	0.264	0.199	0.351	0.184
	(0.064)	(0.065)	(0.152)	(0.165)	(0.277)	(0.299)
Live in Center	0.140**	0.111	0.268	0.208	0.377	0.230
	(0.068)	(0.068)	(0.163)	(0.167)	(0.287)	(0.297)
Intercept	-0.794***	-1.568	-0.024	-0.747	-0.750	-2.441*
	(0.257)	(0.370)	(0.699)	(1.020)	(0.800)	(1.393)
N	1484	1484	344	344	152	152
R^2	0.011		0.057	0.048	0.099	0.036
p-value of Hansen's J test	-	0.970	-	0.827	-	-
p-value of GMM C statistic	-	0.006	-	0.317	-	0.169

Note: Risky assets refer to risky financial assets-i.e. firm or bank bonds, investment funds, stock shares and other financial assets, excluding business equity and real estate. Saving is the sum of market value of the riskfree assets and risky financial assets. Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

For 2010 data, Table 5.7 delivers similar information, and confirms that the coefficient of log saving is not significantly different from zero. The difference is that in column (2), the effect of saving on risky asset share is still significant after controlling for endogeneity; in column (4) the saving level does not cause endogeneity among the sample population who reports their beliefs, and the coefficient of saving is not statistically different from zero for this population. The conclusion from these two tables is that risky asset share is hardly variant to the saving level, and this finding is consistent with the prediction of the model.

Table 5.8 and Table 5.9 investigate whether household saving rate is invariant to wealth level as predicted by the model, using 2008 and 2010 data respectively. For both tables, in columns (1), (3) and (5), I run OLS without controlling for endogeneity caused by wealth level, in columns (2), (4) and (6), I instrument household wealth with household net annual income and/or value of durable goods. The sample in columns (1) and (2) includes households with risky assets, in columns (3) and (4) includes households with risky assets and reporting their belief, in columns (5) and (6) includes households with risky assets and positive excess return expectation. The dependent variable is the log of saving rate, and the independent variables include the log of household wealth and other controls. For 2008 data in Table 5.8, columns from (1) to (4) show that wealth level has a significant effect on saving rate even after controlling for the effect of household expectation and endogeneity of household wealth level: a 1% increase in wealth will lead to an increase of 0.201% and 0.135% in saving rate for households without and with reporting their expectation respectively, which is inconsistent with constant relative risk and ambiguity aversion model. I turn to the households expecting positive excess return, which are most relevant to the analysis. Column (6) shows that the coefficient of wealth is not statistically different from zero once I instrument wealth level with net annual income and the value of durable goods, and the magnitude of the coefficient is quite small. The same conclusion holds for the 2010 survey as confirmed by the regression result in Table 5.9. From cross-sectional regression, whether saving rate is invariant to wealth level is inconclusive, depending on the population I focus on. For households with risky assets and expecting a positive equity premium, who are likely to be both risk averse and ambiguity averse, the saving rate is invariant to wealth level; but for the general population, this is not true.

Table 5.8: Cross-sectional test of saving rate invariant to wealth SHIW 2008

	Saving rate 2008					
	(1)	(2)	(3)	(4)	(5)	(6)
logwealth2008	0.471***	0.201***	0.433***	0.135***	0.392***	0.035
	(0.022)	(0.034)	(0.030)	(0.049)	(0.045)	(0.065)
Theta	-	-	0.022	0.096	0.087	0.253
			(0.086)	(0.094)	(0.306)	(0.342)
Sigma	-	-	0.041	0.059	0.182	0.119
			(0.164)	(0.174)	(0.221)	(0.242)
No.household members	-0.055***	-0.056***	-0.049**	-0.041	-0.058	-0.068*
	(0.022)	(0.017)	(0.025)	(0.026)	(0.041)	(0.041)
Age	-0.001	0.003*	0.001	0.008***	0.0001	0.009**
	(0.001)	(0.002)	(0.002)	(0.003)	(0.003)	(0.004)
No.income earner	-0.018	0.019	-0.014	0.031	0.042	0.085
	(0.018)	(0.020)	(0.029)	(0.033)	(0.041)	(0.044)
Male	-0.002	0.042	-0.022	0.035	-0.006	0.015
	(0.025)	(0.028)	(0.040)	(0.047)	(0.065)	(0.069)
Secondary	-0.067	0.075	-0.176***	-0.051	-0.180*	-0.064
	(0.041)	(0.048)	(0.061)	(0.069)	(0.107)	(0.105)
Uni.or more	-0.163***	0.101	-0.262***	-0.003	-0.277**	-0.000
	(0.050)	(0.063)	(0.072)	(0.084)	(0.120)	(0.125)
Employee	-0.056*	-0.039	0.021	0.050	0.005	0.067
	(0.032)	(0.035)	(0.044)	(0.052)	(0.067)	(0.084)
Self-employed	-0.085**	-0.010	0.029	0.132**	0.082	0.234***
	(0.039)	(0.043)	(0.046)	(0.056)	(0.062)	(0.086)
Live in North	-0.056	-0.017	-0.026	0.076	0.003	0.060
	(0.036)	(0.039)	(0.073)	(0.079)	(0.089)	(0.096)
Live in Center	-0.140***	-0.142***	-0.138*	-0.060	-0.125	-0.116
	(0.041)	(0.045)	(0.082)	(0.088)	(0.107)	(0.116)
Intercept	-5.398***	-2.905***	-5.063***	-2.456***	-4.642***	-1.345**
	(0.247)	(0.354)	(0.340)	(0.475)	(0.559)	(0.664)
N	1291	1291	528	528	259	259
R^2	0.454	0.336	0.463	0.306	0.412	0.186
p-value of Hansen's J test	-	0.161	-	-	-	0.115
p-value of GMM C statistic	-	0.000	-	0.000	-	0.000

Note: Saving is the sum of market value of the riskfree assets and risky financial assets. Wealth is the sum of saving and nondurable consumption expenditure. Saving rate is the ratio of saving over wealth. Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

Table 5.9: Cross-sectional test of saving rate invariant to wealth SHIW 2010

	Saving rate 2010					
	(1)	(2)	(3)	(4)	(5)	(6)
logwealth2010	0.400***	0.148***	0.419***	0.106*	0.387***	-0.030
	(0.016)	(0.027)	(0.035)	(0.005)	(0.046)	(0.094)
Theta	-	-	-0.319**	-0.156	-0.672	-0.320
			(0.161)	(0.168)	(0.477)	(0.598)
Sigma	-	-	-0.473	-0.444	-0.932***	-0.876*
			(0.295)	(0.306)	(0.357)	(0.467)
No.household members	-0.046***	-0.031**	-0.056**	-0.060**	-0.075**	-0.092*
	(0.012)	(0.013)	(0.023)	(0.029)	(0.034)	(0.048)
Age	-0.0008	0.004***	-0.006**	-0.0005	-0.006	0.004
	(0.001)	(0.001)	(0.003)	(0.003)	(0.004)	(0.004)
No.income earner	-0.029**	-0.016	-0.009	0.021	0.018	0.053
	(0.015)	(0.016)	(0.025)	(0.031)	(0.040)	(0.054)
Male	-0.002	0.022	0.042	0.094**	0.009	0.089
	(0.020)	(0.022)	(0.040)	(0.046)	(0.074)	(0.085)
Secondary	-0.075**	0.079**	-0.214***	-0.039	-0.169*	0.078
	(0.032)	(0.038)	(0.050)	(0.062)	(0.100)	(0.122)
Uni.or more	-0.175***	0.091*	-0.303***	-0.034	-0.285**	0.122
	(0.039)	(0.048)	(0.059)	(0.073)	(0.133)	(0.138)
Employee	-0.054*	-0.058*	-0.066	-0.067	-0.048	-0.003
	(0.029)	(0.031)	(0.057)	(0.063)	(0.091)	(0.105)
Self-employed	-0.080***	-0.044	-0.165***	-0.095	-0.165*	-0.011
	(0.030)	(0.034)	(0.064)	(0.079)	(0.098)	(0.126)
Live in North	-0.080***	0.002	-0.007	0.169**	-0.061	0.205
	(0.025)	(0.027)	(0.072)	(0.083)	(0.122)	(0.140)
Live in Center	-0.097***	-0.003	-0.006	0.169*	-0.054	0.181
	(0.027)	(0.031)	(0.076)	(0.087)	(0.137)	(0.155)
Intercept	-4.640	-2.383***	-4.481***	-1.667***	-4.035***	-0.459
	(0.172)	(0.270)	(0.312)	(0.537)	(0.376)	(0.945)
N	1484	1484	344	344	152	152
R^2	0.445	0.297	0.491	0.255	0.483	0.0085
p-value of Hansen's J test	-	0.000	-	0.130	-	0.664
p-value of GMM C statistic	-	0.000	-	0.000	-	0.000

Note: Saving is the sum of market value of the riskfree assets and risky financial assets. Wealth is the sum of saving and nondurable consumption expenditure. Saving rate is the ratio of saving over wealth. Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

Since I have a rotating panel, I can take advantage of the time variation in panel data: the first-difference can eliminate the unobservable and constant variables. Out of 4621 panel households, 593 households report that they hold risky assets in both surveys. Table 5.10 investigates whether risky asset share change is related to saving change: the dependent variable is change of log risky asset share, and the parameter of interest is the coefficient of log saving change. In columns (1) and (3), the regression is run without considering endogeneity caused by saving, in columns (2) and (4), saving change is instrumented by net annual income change and change in the value of durable goods. Compared with columns (1) and (2), columns (3) and (4) control for the effect of household expectation. Table 5.10 shows that saving change can not explain the change of risky asset share, and its effect is not different from zero. The endogeneity of saving does not seem to be serious. Table 5.11 reports the effect of wealth change in the change of household saving rate. As shown in column (1) and (3), the effect of wealth is still significant after first difference. When I instrument wealth change by the change of net annual income, the significant effect disappears.

Table 5.10: Panel test of risky asset share invariant to saving

	Risky asset share change			
	(1)	(2)	(3)	(4)
Δ logsaving	-0.056	0.215	-0.205	-0.879
	(0.048)	(0.201)	(0.142)	(0.649)
Δ theta	-	-	0.213	0.458
			(0.433)	(0.552)
Δ sigma	-	-	0.768	0.774
			(0.693)	(0.682)
Intercept	0.089***	0.056	0.043	0.191
	(0.031)	(0.037)	(0.084)	(0.166)
N	593	593	81	81
R^2	0.004		0.067	
p-value of Hansen's J test	-		-	
p-value of GMM C statistic	-	0.140	-	0.146

Note: Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

Table 5.11: Panel test of saving rate invariant to wealth

	Saving rate change			
	(1)	(2)	(3)	(4)
Δ logwealth	0.432***	0.161	0.431***	0.160
	(0.036)	(0.105)	(0.075)	(0.218)
Δ theta	-	-	-0.062	0.019
			(0.130)	(0.161)
Δ sigma	-	-	-0.048	-0.038
			(0.305)	(0.340)
Intercept	-0.017	0.010	-0.039	0.010
	(0.032)	(0.015)	(0.032)	(0.046)
N	593	593	81	81
R^2	0.350	0.213	0.382	0.232
p-value of Hansen's J test	-		-	
p-value of GMM C statistic	-	0.003	-	0.257

Note: Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

To sum up, the analysis in this section shows that risky asset share is invariant to saving level, and saving rate is invariant to wealth level, which confirms the predictions of the model. I consider constant relative risk aversion and ambiguity aversion as a good first approximation.

5.5.3 Recovering individual risk and ambiguity aversion

Recovery based on panel data

Once I have recovered individual belief and confirmed the assumption of constant relative risk and ambiguity aversion, time preference β, relative risk aversion ρ, and relative ambiguity aversion A can be uniquely recovered according to Proposition 11. The recovery result in Proposition 11 requires a special panel dataset, where I can observe individual zero and non-zero asset demand under the same return of riskfree asset across time. Assuming that the return of risk free asset is invariant between 2008 and 2010, I have 148 such observations. Note that the recovery argument in Proposition 11 only applies to an individual being both risk averse and ambiguity averse, since the recovery argument is based on the first order conditions, a necessary and sufficient condition for utility maximization when the utility function is concave, which holds for both risk aversion and ambiguity aversion. From Lemma 6 it is known that a necessary condition for risk and ambiguity averse individuals to hold risky assets is that the simple excess return of risky asset is positive. As shown in the above analysis, constant relative risk aversion and ambiguity aversion is a good approximation for such a population. So I only recover preference of individuals who expect simple excess return to be positive. Among these 148 observations, I have 65 observations with positive simple excess return. For the remaining 83 observations (56%), individuals hold risky assets though they think the simple excess return is negative, so they must be either risk loving or ambiguity loving. A large proportion of people being either risk loving or ambiguity loving is also identified by experimental research Wakker (2010). However, my identification framework can not recover preference of such individuals, and it is not of my interest.

Table 5.12: Recovered risk and ambiguity aversions based on panel data

	Time preference β	**Re. risk aversion** ρ	**Re. ambiguity aversion** A
5%	0.964	0.001	0.296
10%	0.977	0.002	0.385
25%	0.992	0.006	0.777
50%	1.010	0.012	1.581
75%	1.016	0.033	3.958
95%	1.023	0.080	16.383
Min	0.942	0.0003	0.114
Max	1.107	0.114	25.415
Mean	1.004	0.023	3.578
Std. Dev.	0.024	0.026	5.243
Observation	47	47	47

Note: The above recovery assumes the riskfree asset return is invariant across two-year period.

Table 5.12 reports the recovered time preference β, relative risk aversion ρ and relative ambiguity aversion A based on 47 panel observations assuming invariant interest rate. The mean of time preference is 1.004 and the median is 1.010, which is quite consistent with existing empirical research. One important feature is that the heterogeneity of time preference across individuals is very moderate, with standard deviation being 0.024. In the following part, when I use cross-sectional data to recover individual risk aversion and ambiguity aversion, I expect that assuming time preference to be homogeneous across individuals would be without much loss of generality.

Table 5.12 shows that the mean of recovered relative risk aversion is 0.023. It is much smaller than other existing research based on micro data, which only estimates relative risk aversion without considering ambiguity aversion, and suggests that relative risk aversion is larger than 1.[8] Although the magnitude is small, the relative risk aversion is different from 0. The t test rejects risk neutral assumption at significance level 1%. Besides, the risk aversion is heterogeneous across individuals, with standard deviation 0.047.

The parameter which interests me most is relative ambiguity aversion, since current research gives very little evidence. Table 5.12 shows the mean of relative risk aversion is 3.578, with minimum 0.114 and maximum 25.415. The heterogeneity of ambiguity aversion is quite large with a standard deviation of 5.243. The t test rejects the hypothesis that relative ambiguity aversion is 0 at significance level 1%.

Recovery based on cross-sectional data

The data requirement based on Proposition 11 is too stringent, instead, in this section I assume time preference is homogeneous and is set to be $\beta = 1$. The above analysis suggests that it is a reasonable assumption. Then from Corollary 7, relative risk aversion and ambiguity aversion can be uniquely recovered from one observation of individual saving rate and risky asset share in cross-sectional data. In 2008 data, among 1734 households who report consistent beliefs, 528 households (30.5%) holds risky assets. Since I am only interested in recovering individual preference which is both risk averse and ambiguity averse, I concentrate on the sample with positive simple excess return expectations. In 2008 data, out of 528 households, 259

[8] However, the empirical evidence is far from being conclusive, the estimated relative risk aversion ranges from 0.05 Binswanger (1981) to over 1000 Schluter and Mount (1976).

households (49.1%) hold expectations that simple excess return of risky assets is positive. In 2010 data, among 1120 households with consistent belief, 344 households (30.7%) report positive risky assets. 112 out of these 344 households (44.2%) expect positive excess return of risky assets, which my analysis focuses on.

Table 5.13: Recovered risk and ambiguity aversions based on cross-sectional data

	Individual preference in 2008		**Individual preference in 2010**	
	ρ	A	ρ	A
5%	0.001	0.198	0.002	0.131
10%	0.002	0.332	0.003	0.369
25%	0.007	0.792	0.007	0.895
50%	0.019	1.782	0.017	1.349
75%	0.054	3.810	0.034	2.876
95%	0.334	8.355	0.237	6.885
Min	0.0001	0.019	0.0002	0.027
Max	0.861	18.670	0.507	23.966
Mean	0.062	2.90	0.050	2.498
Std. Dev.	0.127	3.070	0.096	3.562
Observation	130	130	67	67

Note: The above recovery assumes homogeneous time preference across individuals.

Table 5.13 reports the recovered relative risk aversion and relative ambiguity aversion based on cross-sectional data from 2008 and 2010 respectively. In 2008, among 259 households who hold the expectation that simple excess return of risky assets is positive, 130 households (50.2%) are both risk averse and ambiguity averse. So out of 528 households holding positive risky assets, around 24.6% are both risk averse and ambiguity averse. In 2010, among 112 households who hold positive excess return expectations, 67 households (59.8%) are both risk averse and ambiguity averse. So out of 344 households holding positive risky assets, around 19.5% are both risk averse and ambiguity averse.

In 2008, the mean of relative risk aversion is 0.062. The t test rejects the hypothesis that relative risk aversion is 0 at the 1% significance level.

The mean of relative ambiguity aversion is 2.90. The t test rejects the hypothesis that relative ambiguity aversion is 0 at the 1% significance level. Preference heterogeneity (both in risk aversion and ambiguity aversion) across households can be seen from the standard deviation (0.127 and 3.070 respectively).

In 2010, the mean of relative risk aversion is 0.050. The t test rejects the hypothesis that relative risk aversion is 0 at significance level 1%. The mean of relative ambiguity aversion is 2.498. The t test rejects the hypothesis that relative ambiguity aversion is 0 at significance level 1%. As in 2008, preference heterogeneity across households is large. The recovery result from cross-sectional data is very similar to the one from panel data, which lessens the concern about the homogeneous time preference assumption.

5.5.4 Testing over-identification from expected utility model

I have performed a test on whether the risky asset share and the saving rate are invariant to (financial) wealth level, and the result gives support to the constant relative risk aversion and relative ambiguity aversion assumptions. However, such a test can not distinguish the ambiguity model from the CRRA (constant relative risk aversion) expected utility model, which also gives such a prediction. Actually there is a large literature on testing the first implication and recovering relative risk aversion based on the risky asset share equation and historical stock return and volatility, see Friend and Blume (1975), Chiappori and Paiella (2011). One way to distinguish these two models is to test the over-identification of risk aversion from the CRRA expected utility model as shown in Lemma 8. If the CRRA assumption holds, the relative risk aversion recovered from the consumption equation and the risky asset share equation, i.e. ρ_κ and ρ_α respectively, should coincide.

I first test the over-identification restriction imposed by the CRRA model using a few panel observations, and the result is reported in Table 5.14. The mean of ρ_α is equal to 2.212, which is consistent with other empirical investigations into relative risk aversion based on the risky asset share. The mean of ρ_κ is 0.025, which is much smaller than ρ_α. The t test rejects the null hypothesis that ρ_α and ρ_κ are equal, casting doubt on the

CRRA model.

Table 5.14: Over-identified risk aversion based on panel data

	Identification from asset share	**Identification from saving rate**
	ρ_α	ρ_κ
Min	0.114	0.0003
Max	13.004	0.114
Mean	2.183	0.023
Std. Dev.	2.686	0.026
Observation	47	47
	P-value of t test: 0.014	

Note: The above recovery assumes the riskfree asset return is invariant across two-year period.

Table 5.15: Over-identified risk aversion based on cross-sectional data

	Identification based on 2008 data		**Identification based on 2010 data**	
	ρ_α	ρ_κ	ρ_α	ρ_κ
5%	0.224	0.001	0.139	0.002
10%	0.332	0.002	0.364	0.003
25%	0.660	0.007	0.762	0.007
50%	1.290	0.020	1.152	0.017
75%	2.527	0.055	2.067	0.033
95%	5.366	0.448	3.916	0.237
Min	0.0001	0.0001	0.037	0.0002
Max	8.110	1	11.946	0.507
Mean	1.838	0.073	1.596	0.049
Std. Dev.	1.641	0.156	1.683	0.095
Observation	133	133	68	68
	P-value of t test: 0.000		**P-value of t test: 0.000**	

Note: The above recovery assumes homogeneous time preference across individuals.

Table 5.15 reports the over-identified relative risk aversion from cross-sectional data from 2008 and 2010 respectively. I assume homogeneous time preference across individuals and assume $\beta = 1$. The over-identified relative risk aversion is quite similar to the one recovered from panel data in Table 5.14. The t test rejects with strong significance the hypothesis that ρ_α and ρ_κ are equal.

Based on the over-identification restriction test and the fact that the recovered relative ambiguity aversion is significantly different from zero, I conclude that the data support the constant relative risk aversion and ambiguity aversion model over the CRRA model.

5.5.5 Analyzing risk aversion and ambiguity aversion

Correlation between risk and ambiguity aversion

One interesting and important question is: are households' risk aversion and ambiguity aversion correlated, and will more risk averse households be more ambiguity averse? In Table 5.16, I run a simple OLS regression of ambiguity aversion on risk aversion. It shows the hypothesis that the coefficient is 0 cannot be rejected, and there is no evidence that these two parameters are significantly correlated.

Table 5.16: Correlation between risk and ambiguity aversions

	Data in 2008	**Data in 2010**
	(1)	(2)
Risk aversion	3.778	12.797
	(2.296)	(12.838)
Intercept	2.269***	4.859***
	(0.297)	(0.863)
N	130	67
R^2	0.024	0.118

Note: Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

Factors affecting risk and ambiguity aversion

As can be seen from the above analysis, the heterogeneity of household preference is quite large. What accounts for such large variation? How are individual preference parameters related to individual characteristics?

Table 5.17: OLS regression of individual preference

	Preference in 2008		**Preference in 2010**	
	(1)	(2)	(3)	(4)
logw2008	-0.013	-0.438	-	-
	(0.014)	(0.554)		
logw2010	-	-	-0.031*	0.184
			(0.018)	(0.472)
No.household members	0.026	0.934*	-0.010	0.093
	(0.017)	(0.553)	(0.010)	(0.504)
Age	-0.002	0.015	-0.001	-0.133
	(0.001)	(0.030)	(0.002)	(0.092)
No.income eaner	0.005	-0.168	0.019	-0.093
	(0.024)	(0.510)	(0.016)	(0.515)
Male	0.015	1.547***	0.045*	-0.426
	(0.041)	(0.478)	(0.025)	(1.414)
Married	-0.074	-0.275	0.022	0.877
	(0.050)	(0.664)	(0.033)	(1.293)
Separated	-0.044	0.623	-0.030	0.376
	(0.044)	(0.697)	(0.041)	(1.361)
Widowed	0.036	0.696	0.013	2.851
	(0.087)	(1.009)	(0.069)	(3.500)
Secondary	0.052	-0.909	0.030	-5.086**
	(0.036)	(0.813)	(0.049)	(2.276)
Uni.or more	0.071	-0.075	0.040	-3.277*
	(0.038)	(1.044)	(0.036)	(1.858)
Employee	-0.045	0.070	-0.045	-2.657
	(0.037)	(0.811)	(0.049)	(1.796)
Self-employed	-0.034	0.778	-0.071	-2.669
	(0.033)	(0.822)	(0.046)	(2.073)
Live in North	-0.047	-0.215	0.110	0.521
	(0.087)	(1.164)	(0.061)	(2.595)
Live in Center	-0.026	-0.889	0.071	0.042
	(0.090)	(1.178)	(0.038)	(2.259)
Intercept	0.171	4.500	0.312	12.709**
	(0.256)	(6.195)	(0.180)	(6.083)
N	130	130	67	67
Pseudo R^2	0.111	0.196	0.184	0.200

Note: Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

Table 5.17 reports the OLS regression of household preference (both relative risk aversion and ambiguity aversion) on individual characteristics. In columns (1) and (3), the dependent variable is relative risk aversion, and in columns (2) and (4), the dependent variable is relative ambiguity aversion. In 2008, male household heads are more ambiguity averse; but this correlation is not significant in 2010. The result shows that large variation of risk and ambiguity aversion across households cannot be explained by observable household characteristics: their effect is negligible. Noticeably, wealth level is neither correlated to relative risk aversion nor correlated to relative ambiguity aversion, presenting evidence in favor of the constant relative risk and ambiguity aversion assumption.

Risky asset share holding

How will individual risk aversion and ambiguity aversion affect consumption and portfolio choice? In the theoretical model in Section 5.2, I examine the qualitative effect of risk aversion and ambiguity aversion. In this section, I present evidence of the quantitative effect.

In Table 5.18, I run a regression of the risky asset holdings on household preference, expectation, and other observable characteristics. In columns (1) and (3), I run simple OLS, and in columns (2) and (4), I instrument household wealth by net annual income. Wealth has a significant positive effect on the risky asset holding in both 2008 and 2010. Risk aversion is positively related to the risky asset holding in 2008, but such an effect is not significant in 2010. In contrast, ambiguity aversion is significantly negatively correlated with risky asset holding across time. The expected mean of the risky asset return has a significant positive effect on risky asset holdings, however the effect of ambiguity is not significant. All these results are consistent with the predictions of the model.

Table 5.18: OLS regression of risky asset holding

	Risky asset in 2008		**Risky asset in 2010**	
	(1)	(2)	(3)	(4)
logw2008	1.637***	1.302***	-	-
	(0.096)	(0.127)		
logw2010	-	-	1.501***	0.744*
			(0.118)	(0.419)
Risk aversion	0.858**	0.694**	-0.128	-2.288
	(0.359)	(0.326)	(0.807)	(1.932)
Ambiguity aversion	-0.121***	-0.134***	-0.094***	-0.079***
	(0.024)	(0.025)	(0.022)	(0.030)
Theta	2.104***	2.975***	3.612***	8.137**
	(0.783)	(0.856)	(1.264)	(3.549)
Sigma	-0.023	0.079	-3.089***	-0.596
	(0.473)	(0.468)	(1.087)	(1.508)
No.household members	-0.084	-0.102	-0.015	0.046
	(0.086)	(0.077)	(0.056)	(0.091)
Age	-0.001	0.003	-0.009	0.013
	(0.007)	(0.007)	(0.008)	(0.022)
No.income eaner	0.046	0.060	-0.015	-0.115
	(0.105)	(0.098)	(0.091)	(0.193)
Male	-0.126	-0.020	-0.093	-0.221
	(0.140)	(0.141)	(0.195)	(0.238)
Secondary	-0.387	-0.456*	-0.565*	3.161
	(0.256)	(0.234)	(0.289)	(6.689)
Uni.or more	-0.460*	-0.357	-0.706**	3.376
	(0.276)	(0.254)	(0.329)	(6.913)
Employee	0.002	0.095	0.030	-0.310
	(0.188)	(0.192)	(0.230)	(0.486)
Self-employed	0.193	0.368**	-0.183	-0.035
	(0.154)	(0.163)	(0.205)	(0.374)
Intercept	-7.246***	-3.795***	-4.880***	-1.985
	(1.203)	(1.452)	(1.045)	(4.445)
N	130	130	67	67
Pseudo R^2	0.832	0.815	0.919	0.744
p-value of Hansen's J test	-	0.415	-	1.000
p-value of GMM C statistic	-	0.007	-	0.001

Note: Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

Table 5.19: OLS regression of first period consumption

	Consumption in 2008		**Consumption in 2010**	
	(1)	(2)	(3)	(4)
logw2008	0.401***	0.075***	-	-
	(0.072)	(0.105)		
logw2010	-	-	0.234**	1.732
			(0.088)	(6.467)
Risk aversion	0.664***	0.775***	0.642	3.926
	(0.193)	(0.182)	(0.758)	(13.529)
Ambiguity aversion	-0.001	0.011	0.003	0.021
	(0.012)	(0.013)	(0.015)	(0.147)
Theta	-0.936**	-1.711***	-1.524	-9.421
	(0.419)	(0.487)	(1.002)	(34.135)
Sigma	-0.362	-0.444	1.226	-4.172
	(0.260)	(0.300)	(0.748)	(22.403)
No.household members	0.124**	0.148***	0.077*	0.017
	(0.051)	(0.046)	(0.042)	(0.284)
Age	-0.001	-0.006	0.013**	-0.002
	(0.004)	(0.005)	(0.006)	(0.037)
No.income eaner	-0.013	-0.041	0.048	-0.003
	(0.052)	(0.052)	(0.066)	(0.329)
Male	0.015	-0.107	0.040	0.467
	(0.076)	(0.091)	(0.144)	(2.052)
Secondary	0.064	0.146	0.369*	1.268
	(0.112)	(0.112)	(0.200)	(6.802)
Uni.or more	0.066	-0.011	0.648***	0.846
	(0.121)	(0.133)	(0.195)	(3.702)
Employee	-0.041	-0.154	-0.099	0.518
	(0.090)	(0.119)	(0.163)	(3.154)
Self-employed	-0.140	-0.302***	0.030	-0.471
	(0.097)	(0.111)	(0.173)	(1.828)
Intercept	-7.246***	2.253**	5.668***	-9.817
	(1.203)	(1.042)	(0.873)	(772.881)
N	130	130	67	67
Pseudo R^2	0.550	0.038	0.625	
p-value of Hansen's J test	-	0.384	-	1.00
p-value of GMM C statistic	-	0.001	-	0.406

Note: Robust standard errors in parentheses. *significant at 10%, **significant at 5%, ***significant at 1%.

In Table 5.19, I present evidence of the effect of household preference on consumption decisions. Risk aversion has a significantly positive effect on the consumption level in 2008; in contrast to the risky asset holding case, ambiguity aversion does not show any significant effect on consumption. The expected mean of the risky asset return is negatively related to consumption in 2008, but ambiguity has no significant effect in either period. Household wealth level and the number of household members is positively correlated with consumption level in both periods. Marital status and living region have significant effect in 2008, but not in 2010.

So risk aversion governs both consumption and portfolio choice, however, ambiguity aversion mainly affects the allocation of savings to the risky assets.

5.6 Robustness checks

In this section, I show results from robustness check. To deal with the concern of mis-specification of parameters R, σ^2 and β, I examine how sensitive are the recovered preference parameters to the specification of these parameters.

Alternative parameter specifications In the baseline results reported in Section 5.5, I assume households know the riskfree asset return R and the variance of log risky asset return σ^2. I take the value of R to be 0.983, the value of σ^2 to be 0.02. When I do recovery using cross-sectional data, I also assume the homogeneous time preference β to be 1. I check robustness of the recovery results under the following alternative specifications: $R = 1$ and $R = 1.01$; $\sigma^2 = 0.04$ and $\sigma^2 = 0.06$; $\beta = 0.99$ and $\beta = 0.98$.

The robustness check based on panel data shows that the recovered preference parameters−time preference β, relative risk aversion ρ and relative ambiguity aversion A are sensitive to belief parameters R and σ^2. Given $\sigma^2 = 0.02$, when $R = 1$, the means (standard deviations) of β, ρ and A are 0.989 (0.019), 0.019 (0.020) and 2.617 (3.093); when $R = 1.01$, the means (standard deviations) of β, ρ and A are 0.981 (0.016), 0.016 (0.017) and 1.736 (1.771). Given $R = 1$, when $\sigma^2 = 0.04$, the means (standard deviations) of β, ρ and A are 0.988 (0.020), 0.021 (0.021) and 7.745 (15.559);

when $\sigma^2 = 0.06$, the means (standard deviations) of β, ρ and A are 0.984 (0.017), 0.022 (0.021) and 3.319 (3.155).

The robustness check based on cross-sectional data shows that the recovered relative risk aversion ρ and relative ambiguity aversion A are sensitive to belief parameters R and σ^2, but less sensitive to time preference parameter β. In 2008 data, given $\sigma^2 = 0.02$ and $R = 0.983$, when $\beta = 0.99$, the means (standard deviations) of ρ and A are 0.078 (0.168) and 3.046 (3.358); when $\beta = 0.98$, the means (standard deviations) of ρ and A are 0.068 (0.140) and 2.927 (3.727). Given $\sigma^2 = 0.02$ and $\beta = 1$, when $R = 1$, the means (standard deviations) of ρ and A are 0.060 (0.122) and 3.815 (6.203); when $R = 1.01$, the means (standard deviations) of ρ and A are 0.072 (0.127) and 3.144 (5.202). Given $R = 1$ and $\beta = 1$, when $\sigma^2 = 0.04$, the means (standard deviations) of ρ and A are 0.065 (0.127) and 7.431 (12.444); when $\sigma^2 = 0.06$, the means (standard deviations) of ρ and A are 0.075 (0.137) and 9.115 (14.733). A robustness check from the 2010 data delivers similar results.

The basic conclusion from the robustness check exercise is that the exact magnitude of households' relative risk aversion and relative ambiguity averion depends on how precise household belief is; however, some robust results are preserved: the recovered preferences display considerable heterogeneity, the average relative risk aversion is less than 1, and the average relative ambiguity aversion is 3 or larger.

5.7 Conclusion

5.7.1 Summary of results

Despite growing interest in modelling ambiguity aversion in decision theory, and the widespread use of ambiguity aversion models in economics and finance, the problem of identification has not been addressed until recently. In this chapter, under some reasonable assumptions about the parametric form of ambiguity preference and individual belief, I examine the shape of individual ambiguity preference using data from the Italian household survey. The findings of this chapter can be summarized as follows.

The recovered household expectations are heterogeneous and pessimistic. The expected mean of the stock market returns is barely positive, and much lower than historical realization. The variance of expected stock returns is much higher than historical volatility, which implies much ambiguity under my model. The result here is qualitatively consistent with existing evidence.

The evidence supports that constant relative risk aversion and ambiguity aversion can be a good approximation. The saving rate out of wealth is invariant to wealth level, and the risky asset share out of saving is invariant to the saving level. The data, with high significance, rejects other functional forms like constant absolute risk aversion and/or constant relative ambiguity aversion, which imply that the value of the risky asset is invariant with the saving level. Constant relative ambiguity aversion has one desirable property: it is independent of the utility unit attached to risk preference.

The recovered average relative risk aversion is much smaller than 1 and the recovered average relative ambiguity aversion is about 3 or larger. I do the recovery, firstly, using a special panel dataset assuming that the risk-free interest rate constant across time. The recovered time preference is fairly homogeneous across individuals, and consistent with existing micro-evidence. Without loss of generality, I assume the same time preference across individuals, and use cross-sectional data on consumption, saving and portfolio choice to recover individual relative risk aversion and ambiguity aversion. The recovered relative risk aversion is smaller than the one estimated when only considering risk (usually it is believed that relative risk aversion is between 1 and 2). Currently there is no comparable evidence on the magnitude of relative ambiguity aversion.

The chapter also distinguishes two models—the subjective expected utility model and the smooth ambiguity aversion model. If individuals are subjective expected utility maximizers, their consumption and portfolio choice will put a strong over-identification restriction on relative risk aversion. The data rejects such a restriction at a high significance level. The evidence that the recovered relative ambiguity aversion is significantly different from 0 also gives support to smooth ambiguity aversion model.

This chapter provides further analysis of individual risk aversion and ambiguity aversion. Relative risk aversion is not significantly related to relative ambiguity aversion, and the large variation of risk aversion and ambiguity aversion can hardly be explained by household characteristics.

Quantitatively, risk aversion and ambiguity aversion have a significant effect on consumption and risky asset holding, however, they play different roles.

5.7.2 Future research

In this chapter, the model is two-period, which means the individual is doing short-horizon planning. Such an assumption can be restrictive, since the individual is believed to be fully rational and will take the future into consideration when making decisions. One area for future research is to build a long-horizon model (finite or infinite), and to derive the corresponding Euler equation, based on which to estimate preference parameters. Another restrictive assumption I make is that the only risk (and ambiguity) comes from asset returns, and there is no background risk. Background risk will bring additional difficulties for identification, which is still an open question.

The data I use is not fully satisfying in the sense that there is large attrition with respect to non-reporting expectations about stock market performance. How this will influence our result is not clear. This could be captured in a future survey with a specific module to elicit out reliable individual ambiguous beliefs.

Appendix A. Identification when $\rho > 1$

In Section 5.2 and Section 5.3 of this chapter, I solve the individual optimization problem and prove the identification under the assumption that relative risk aversion ρ is less than 1. In this part, I establish the corresponding results when the relative risk aversion is greater than 1.

Suppose

$$u(c) = \frac{c^{1-\rho}}{1-\rho}, \tag{A1}$$

where $\rho > 1$, and

$$\phi(u) = -\frac{(-u)^{1+A}}{1+A}, \tag{A2}$$

then

$$\phi^{-1} = -[-(1+A)\phi]^{\frac{1}{1+A}}. \tag{A3}$$

Consumers will solve the problem:

$$\max_{\{c_0,\alpha\}} \; u(c_0)+\beta\phi^{-1}\Big\{E_{\tilde{\boldsymbol{\mu}}}\phi\Big[E_{\tilde{\boldsymbol{\nu}}}u\Big(\big(R+(\tilde{\boldsymbol{\nu}}-R)\alpha\big)(w-c_0)\Big)\Big]\Big\}. \quad \text{(A4)}$$

The log-normal approximation of portfolio return would be

$$\ln\big(R+(\tilde{\boldsymbol{\nu}}-R)\alpha\big)\approx r+\alpha\big(\ln(\tilde{\boldsymbol{\nu}})-\ln(R)\big)+\alpha\frac{\sigma^2+\sigma_0^2}{2}-\alpha^2\frac{\sigma^2+\sigma_0^2}{2}. \quad \text{(A5)}$$

Then the expected second-period utility for one realization μ of $\tilde{\boldsymbol{\mu}}$:

$$E_{\tilde{\boldsymbol{\nu}}}u(c_1)=\frac{(I-c_0)^{1-\rho}}{1-\rho}\exp\Big(F(\alpha)\Big)\exp\Big[(1-\rho)\alpha(\mu-r)\Big], \quad \text{(A6)}$$

where

$$F(\alpha)=(1-\rho)(r+\alpha\frac{\sigma^2+\sigma_0^2}{2}-\alpha^2\frac{\rho\sigma^2+\sigma_0^2}{2}). \quad \text{(A7)}$$

Determination of α

$$E_{\tilde{\boldsymbol{\mu}}}[\phi(E_{\tilde{\boldsymbol{\nu}}}u(c_1))]=\frac{-[\frac{(I-c_0)^{1-\rho}}{\rho-1}]^{1+A}}{1+A}G(\alpha)E_{\tilde{\boldsymbol{\mu}}}\exp\Big[(1+A)(1-\rho)\alpha\tilde{\boldsymbol{\mu}}\Big], \quad \text{(A8)}$$

where

$$G(\alpha)=\exp\Big((1+A)(1-\rho)(r+\alpha\frac{\sigma^2+\sigma_0^2}{2}-\alpha^2\frac{\rho\sigma^2+\sigma_0^2}{2}-\alpha r)\Big). \quad \text{(A9)}$$

Given $\tilde{\boldsymbol{\mu}}\sim N(\theta,\sigma_0^2)$,

$$(1+A)(1-\rho)\alpha\tilde{\boldsymbol{\mu}}\sim N\Big((1+A)(1-\rho)\alpha\theta,(1+A)^2(1-\rho)^2\alpha^2\sigma_0^2\Big). \quad \text{(A10)}$$

Then

$$E_{\tilde{\boldsymbol{\mu}}}\exp\Big((1+A)(1-\rho)\alpha\tilde{\boldsymbol{\mu}}\Big)=\exp\Big((1+A)(1-\rho)\alpha\theta+\alpha^2\frac{(1+A)^2(1-\rho)^2\sigma_0^2}{2}\Big). \quad \text{(A11)}$$

Therefore,

$$\phi^{-1}\Big\{E_{\tilde{\boldsymbol{\mu}}}\Big[\phi\Big(E_{\tilde{\boldsymbol{\nu}}}u(c_1)\Big)\Big]\Big\}=\frac{(I-c_0)^{1-\rho}}{1-\rho}H(\alpha)\exp\Big(\alpha^2\frac{(1+A)(1-\rho)^2\sigma_0^2}{2}\Big), \quad \text{(A12)}$$

where

$$H(\alpha) = \exp\left[(1-\rho)\left(r + \alpha(\theta - r) + \alpha\frac{\sigma^2+\sigma_0^2}{2} - \alpha^2\frac{\rho\sigma^2+\sigma_0^2}{2}\right)\right]. \quad \text{(A13)}$$

Since in the individual optimization problem, portfolio choice and first-period consumption choice are uncorrelated, without loss of generality, α can be determined by solving the following maximization problem:

$$\max_{\alpha\in[0,1]} \phi^{-1}\left\{E_{\tilde{\boldsymbol{\mu}}}\left[\phi\left(E_{\tilde{\boldsymbol{\nu}}}u(c_1)\right)\right]\right\}. \quad \text{(A14)}$$

First order condition:

$$\theta - r + \frac{\sigma^2+\sigma_0^2}{2} - \left((\rho\sigma^2+\sigma_0^2) - (1+A)(1-\rho)\sigma_0^2\right)\alpha = 0. \quad \text{(A15)}$$

Then the optimal solution is

$$\alpha = \frac{\theta - r + \frac{\sigma^2+\sigma_0^2}{2}}{\rho(\sigma^2+\sigma_0^2) + (\rho-1)A\sigma_0^2}. \quad \text{(A16)}$$

Determination of c_0

$$\begin{aligned}\max_{\{c_0,\alpha\}} u(c_0) + \beta\phi^{-1}&\left\{E_{\tilde{\boldsymbol{\mu}}}\left[\phi\left(E_{\tilde{\boldsymbol{\nu}}}u(c_1)\right)\right]\right\} \\ &= \frac{c_0^{1-\rho}}{1-\rho} + \beta\frac{(I-c_0)^{1-\rho}}{1-\rho}\exp\Big((1-\rho)K(\alpha)\Big),\end{aligned} \quad \text{(A17)}$$

where

$$K(\alpha) = r + \alpha(\theta - r) + \alpha\frac{\sigma^2+\sigma_0^2}{2} - \alpha^2\frac{\rho\sigma^2+\sigma_0^2}{2} + \alpha^2\frac{(1+A)(1-\rho)^2\sigma_0^2}{2}. \quad \text{(A18)}$$

F.O.C w.r.t c_0,

$$c_0^{-\rho} = \beta(I-c_0)^{-\rho}\exp\Big((1-\rho)H(\alpha)\Big). \quad \text{(A19)}$$

It gives

$$c_0 = \kappa w, \quad \text{(A20)}$$

where

$$\kappa = \left\{1 + \beta^{\frac{1}{\rho}} \exp\left[\frac{(1-\rho)}{\rho}\left(r + \frac{(\theta - r + \frac{\sigma^2+\sigma_0^2}{2})^2}{2\big(\rho(\sigma^2+\sigma_0^2) + (\rho-1)A\sigma_0^2\big)}\right)\right]\right\}^{-1}. \quad \text{(A21)}$$

Identification

Lemma 9. *Assume that the individual has constant relative risk aversion and constant relative ambiguity aversion. Assume the risky asset return is log-normal with ambiguous mean being normally distributed. Suppose*

1. *at time s, with asset returns $(\tilde{\boldsymbol{\nu}}_s, R)$, the individual has consumption rate κ^s, and only invest in the riskfree asset;*

2. *at time t, with asset return $(\tilde{\boldsymbol{\nu}}_t, R)$, the individual has consumption rate κ^t, and invest α^t of saving in the risky asset.*

Then β, ρ and A can be uniquely identified as

$$\rho = \frac{\frac{\alpha^t(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2})}{2}}{\ln\frac{1-\kappa^t}{\kappa^t} - \ln\frac{1-\kappa^s}{\kappa^s} + \frac{\alpha^t(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2})}{2}}, \quad \text{(A22)}$$

$$\beta = \left(\frac{1-\kappa^s}{\kappa^s}\right)^{\rho} \exp\Big((\rho-1)r\Big), \quad \text{(A23)}$$

$$A = \frac{(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2}) - \alpha^t\rho(\sigma_t^2+\sigma_{0t}^2)}{\alpha^t(\rho-1)\sigma_{0t}^2}. \quad \text{(A24)}$$

Proof. It follows the same argument in the proof of Proposition 11. □

Corollary 9. *Assume that the individual has constant relative risk aversion and constant relative ambiguity aversion. Assume the risky asset return is log-normal with ambiguous mean being normally distributed. Suppose at time t, with asset return $(\tilde{\boldsymbol{\nu}}_t, R)$, the individual has consumption rate κ^t, and invest α^t of saving in the risky asset. If individual time preference β is known, then individual relative risk aversion and ambiguity aversion can* *be uniquely identified as*

$$\rho = \frac{\ln\beta + r + \frac{\alpha^t(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2})}{2}}{\ln\frac{1-\kappa^t}{\kappa^t} + r + \frac{\alpha^t(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2})}{2}}, \tag{A25}$$

$$A = \frac{(\theta_t - r + \frac{\sigma_t^2+\sigma_{0t}^2}{2}) - \alpha^t\rho(\sigma_t^2 + \sigma_{0t}^2)}{\alpha^t(\rho-1)\sigma_{0t}^2}. \tag{A26}$$

Appendix B. Interpretation of relative risk and ambiguity aversion

In this part, I characterize the uncertainty premium in terms of relative (or multiplicative) uncertainty. This characterization will make it clear what relative risk aversion and relative ambiguity aversion measure in the smooth ambiguity model.

Consider a multiplicative uncertainty $\tilde{\boldsymbol{w}} = w_0(1+k\tilde{\boldsymbol{x}}) = w_0(1+\tilde{\boldsymbol{y}})$ with $E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}}\tilde{\boldsymbol{x}} = 0$. Then uncertainty premium $p(k) = e(w_0, u, \phi, k\tilde{\boldsymbol{x}})$ is defined by

$$E_{\boldsymbol{\mu}}\phi\Big(E_{\boldsymbol{\nu}}u\big(w_0(1+k\tilde{x})\big)\Big) = \phi\Big(u\big(w_0 - p(k)\big)\Big). \tag{B1}$$

Then it satisfies

$$p(0) = 0. \tag{B2}$$

Differentiate both sides of equation (B1) with respect to k, the following holds

$$\begin{aligned} &E_{\boldsymbol{\mu}}\phi'\Big(E_{\boldsymbol{\nu}}u\big(w_0(1+k\tilde{\boldsymbol{x}})\big)\Big)E_{\boldsymbol{\nu}}u'\big(w_0(1+k\tilde{\boldsymbol{x}})\big)w_0\tilde{\boldsymbol{x}} \\ &= \phi'\Big(u\big(w_0 - p(k)\big)\Big)u'\big(w_0 - p(k)\big)(-p'(k)). \end{aligned} \tag{B3}$$

At $k = 0$, it becomes

$$\phi'\big(u(w_0)\big)u'(w_0)w_0E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}}\tilde{\boldsymbol{x}} = \phi'(u(w_0))u'(w_0)(-p'(0)). \tag{B4}$$

Since the uncertainty satisfies $E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}}\tilde{\boldsymbol{x}} = 0$, therefore

$$p^{'}(0) = 0. \tag{B5}$$

Further differentiate both sides of equation (B3) with respect to k, it gives

$$\begin{aligned}
&E_{\boldsymbol{\mu}}\phi^{''}\Big(E_{\boldsymbol{\nu}}u\big(w_0(1+k\tilde{\boldsymbol{x}})\big)\Big)\Big(E_{\boldsymbol{\nu}}u^{'}\big(w_0(1+k\tilde{\boldsymbol{x}})\big)w_0\tilde{\boldsymbol{x}}\Big)^2\\
&\quad + E_{\boldsymbol{\mu}}\phi^{'}\Big(E_{\boldsymbol{\nu}}u\big(w_0(1+k\tilde{\boldsymbol{x}})\big)\Big)E_{\boldsymbol{\nu}}u^{''}\big(w_0(1+k\tilde{\boldsymbol{x}})\big)w_0^2\tilde{\boldsymbol{x}}^2\\
=&\phi^{'}\Big(u\big(w_0-p(k)\big)\Big)\Big(u^{'}\big(w_0-p(k)\big)\big(-p^{'}(k)\big)\Big)^2\\
&\quad + \phi^{'}\Big(u\big(w_0-p(k)\big)\Big)u^{''}\big(w_0-p(k)\big)\big(-p^{'}(k)\big)^2\\
&\quad + \phi^{'}\Big(u\big(w_0-p(k)\big)\Big)u^{'}\big(w_0-p(k)\big)\big(-p^{''}(k)\big).
\end{aligned} \tag{B6}$$

At $k = 0$, it becomes

$$\begin{aligned}
&\phi^{''}\big(u(w_0)\big)u^{'2}(w_0)w_0^2E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}}\tilde{\boldsymbol{x}})^2 + \phi^{'}\big(u(w_0)\big)u^{''}(w_0)w_0^2E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}}\tilde{\boldsymbol{x}}^2\\
&= \phi^{'}\big(u(w_0)\big)u^{'}(w_0)\big(-p^{''}(0)\big).
\end{aligned} \tag{B7}$$

Take Taylor expansion of $p(k)$ around $k = 0$, it gives

$$p(k) = p(0) + kp^{'}(0) + \frac{1}{2}k^2p''(0). \tag{B8}$$

Substitute $p(0)$, $p^{'}(0)$ and $p^{''}(0)$ into above equation (B8), it follows

$$\begin{aligned}
\frac{e(w_0,u,\phi,k\tilde{\boldsymbol{x}})}{w_0} &= \frac{p(k)}{w_0}\\
&= \frac{1}{2}k^2[-\frac{\phi^{''}\big(u(w_0)\big)}{\phi^{'}\big(u(w_0)\big)}u^{'}(w_0)w_0E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}}\tilde{\boldsymbol{x}})^2 - \frac{u^{''}(w_0)}{u^{'}(w_0)}w_0E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}}\tilde{\boldsymbol{x}}^2]\\
&= \frac{1}{2}[a^R(w_0,u)u^EE_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}}\tilde{\boldsymbol{y}})^2 + r^R(w_0)E_{\boldsymbol{\mu}}E_{\boldsymbol{\nu}}\tilde{\boldsymbol{y}}^2],
\end{aligned} \tag{B9}$$

where $a^R = -\frac{\phi^{''}\left(u(w_0)\right)}{\phi^{'}\left(u(w_0)\right)}u(w_0)$ is the relative ambiguity aversion, $u^E = \frac{u^{'}(w_0)}{u(w_0)}w_0$ is the elasticity of utility with respect to wealth, and $r^R = -\frac{u^{''}(w_0)}{u^{'}(w_0)}w_0$

is the relative risk aversion.

In the above decomposition of uncertainty premium (B9), the second term $\frac{1}{2}r^R E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}} \tilde{\boldsymbol{y}}^2$ is the risk premium−the share of wealth individual would like to pay to avoid the risk indexed by $E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}} \tilde{\boldsymbol{y}}^2$; the first term $\frac{1}{2}a^R(w_0, u)u^E E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}} \tilde{\boldsymbol{y}})^2$ is the ambiguity premium−the share of wealth individual would like to pay to avoid the dispersion of expected values, and it will becomes 0 if $E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}} \tilde{\boldsymbol{y}})^2 = 0$, i.e. no dispersion of expected value.

The meaning of relative risk aversion and relative ambiguity aversion becomes clear now. Denote the ambiguity premium by $e_a(w_0, u, \phi, k\tilde{\boldsymbol{x}})$, and risk premium by $e_r(w_0, u, \phi, k\tilde{\boldsymbol{x}})$, then

$$r^R = \frac{2e_r(w_0, u, \phi, k\tilde{\boldsymbol{x}})}{E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}} \tilde{\boldsymbol{y}}^2}, \tag{B10}$$

$$a^R = \frac{2e_a(w_0, u, \phi, k\tilde{\boldsymbol{x}})}{u^U E_{\boldsymbol{\mu}}(E_{\boldsymbol{\nu}} \tilde{\boldsymbol{y}})^2}. \tag{B11}$$

More specifically, relative ambiguity aversion measures the share of wealth individual would like to pay for replacing the ambiguous prospect with the purely risky one with mean $E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}} \tilde{\boldsymbol{y}}$ which equals to 0 here, and variance $E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}} \tilde{\boldsymbol{y}}^2$. Relative risk aversion measures the share of wealth individual would like to pay for replacing the purely risk prospect with its expected mean $E_{\boldsymbol{\mu}} E_{\boldsymbol{\nu}} \tilde{\boldsymbol{y}}$.

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